

# Combinatorial restrictions on cell complexes

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Von der Fakultät II – Mathematik und Naturwissenschaften  
der Technischen Universität Berlin  
zur Erlangung des akademischen Grades  
Doktor der Naturwissenschaften  
– Dr. rer. nat. –

genehmigte Dissertation

Promotionsausschuss:

Vorsitzender: Prof. Dr. rer. nat. Stefan Felsner

Berichter: Prof. John M. Sullivan, Ph.D.

Berichter: Prof. Dr. Imre Bárány

Tag der wissenschaftlichen Aussprache: 29.06.2015

Berlin 2015



## Acknowledgements

I am glad to have been part of the Arbeitsgruppe Geometrie at Technische Universität Berlin over the last couple of years. Here I found a great atmosphere with many helpful discussions. I have enjoyed the hospitality of everyone at Arnimallee 2 (Freie Universität Berlin) during my (more or less) weekly visits with many interesting conversations.

John M. Sullivan has been a wonderful advisor who has always made time for me to discuss and help me along with suggestions, remarks, and questions. He has also granted me the freedom to research the topics I find interesting. Pavle Blagojević has introduced me to the methods of equivariant topology and their use in combinatorics and discrete geometry, which have fascinated me ever since. Without his constant encouragement (and coauthorship) the results of Chapter 2 would have not been possible. Frank Lutz has helped to shape my understanding of triangulations in many discussions and is a coauthor for the results in Chapter 3. Günter M. Ziegler's advice and encouragement have been important to me. The results of Chapter 2 are also joint with him. I would like to thank John, Pavle, Frank, and Günter for their constant support. Time and again they have helped me, listened to my ideas, and shared their own ideas, which I truly appreciate.

Raman Sanyal has been an important mentor for which I would like to thank him. Thank you, Imre Bárány for agreeing to be the external reviewer of this dissertation. Some of the results of Chapter 2 and 3 were shaped in particular by discussions with Moritz Firsching and Albert Haase. I would like to thank them for their input.

For three years I was supported by a scholarship of the German Science Foundation DFG via the Berlin Mathematical School, which has enabled me to focus on my research.

Florian Frick, Berlin, April 2015



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# 1 Introduction

This thesis attempts to give partial answers concerning the problem of how combinatorial restrictions affect the topology, geometry, embeddability, and convexity of cell complexes. As toy examples we will briefly review graph planarity and the combinatorial Gauß–Bonnet theorem and discuss some higher-dimensional versions. We will allow ourselves to be somewhat imprecise at times in this introduction, but we add precise statements in subsequent sections.

Kuratowski’s theorem [60] characterizes *planar graphs*, that is, 1-dimensional cell complexes that are embeddable into  $\mathbb{R}^2$ , as those graphs without a subspace homeomorphic to either  $K_5$  or  $K_{3,3}$ . Here  $K_n$  denotes the *complete graph* on  $n$  vertices and  $K_{n,m}$  denotes the *complete bipartite graph* on  $n + m$  vertices with independent sets of size  $n$  and  $m$ . By a theorem of Fáry [37] (whose first proof seems to be due to Wagner [102]), a graph is planar if and only if it admits an embedding into  $\mathbb{R}^2$  with straight edges. For any continuous map from a non-planar graph to  $\mathbb{R}^2$  there are always two vertex-disjoint edges that have a point in common by the Hanani–Tutte theorem; see Tutte [96]. Thus, it suffices to check whether vertex-disjoint edges intersect to test the planarity of a graph.

We will be interested in higher-dimensional analogues of these theorems. Deciding the embeddability of a  $k$ -dimensional complex into  $\mathbb{R}^n$  is in general difficult. For example, it is algorithmically undecidable whether a  $(d-1)$ -dimensional complex is embeddable into  $\mathbb{R}^d$  for  $d \geq 5$  by a result of Matoušek, Tancer, and Wagner [69]. Fáry’s theorem is no longer valid in higher dimensions. It is not in general possible to straighten topological embeddings into  $\mathbb{R}^d$ ,  $d \geq 3$ : there are triangulations of the orientable surface of genus five that do not admit a simplex-wise linear embedding into  $\mathbb{R}^3$ ; see Schewe [83]. For genus  $g \geq 6$  such triangulations were found earlier by Bokowski and Guedes de Oliveira [19], disproving a conjecture of Grünbaum. It is an open question whether triangulations of orientable surfaces of genus  $1 \leq g \leq 4$  exist that cannot be embedded into 3-space in a simplex-wise linear fashion. Earlier, Brehm [22] had already constructed a triangulation of the Möbius strip that is not simplex-wise linear embeddable into  $\mathbb{R}^3$ .

Higher-dimensional versions of the Hanani–Tutte theorem are equivalent to deciding the completeness of the Van Kampen obstruction — a  $\mathbb{Z}/2$ -equivariant cochain capturing the number of intersections of two disjoint facets modulo two. Van Kampen [54] essentially

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introduced this obstruction, although the language of cohomology had not been fully developed at this point. A vanishing of this obstruction is necessary for the embeddability of a  $d$ -complex into  $\mathbb{R}^{2d}$ . Van Kampen and independently Flores [38] used this to show that the  $d$ -skeleton  $\Delta_{2d+2}^{(d)}$  of the  $(2d+2)$ -simplex is not embeddable into  $\mathbb{R}^{2d}$ , and that there is always an intersection of two disjoint faces. This is known as the Van Kampen–Flores theorem. Completeness refers to the vanishing of the obstruction also being sufficient for embeddability of a complex. Van Kampen’s proof of the completeness of his obstruction had a lacuna that was essentially closed by Whitney [105] with what became known as *Whitney trick*. Whitney was concerned with embedding smooth manifolds into Euclidean space, but his reasoning works for general simplicial complexes as pointed out by Shapiro [84]. Freedman, Krushkal, and Teichner [40] show the exact analogue of the Hanani–Tutte theorem for maps from a  $d$ -dimensional simplicial complex into  $\mathbb{R}^{2d}$  for  $d \geq 3$ . Moreover, they show that the Van Kampen obstruction is incomplete for the case  $d = 2$ .

A first step towards a higher-dimensional Kuratowski theorem was taken by Radon [78]. A simple reformulation of his result about intersecting convex hulls of points is that every affine map  $f: \Delta_{d+1} \rightarrow \mathbb{R}^d$  from the  $(d+1)$ -simplex  $\Delta_{d+1}$  to  $\mathbb{R}^d$  identifies points from two disjoint faces of  $\Delta_{d+1}$ . As straightening results are generally wrong in dimensions  $d \geq 3$ , this does not imply the same statement for  $f$  only a continuous map. While Radon’s theorem follows from elementary linear algebra, its topological generalization by Bajmóczy and Bárány [5] requires the use of the Borsuk–Ulam theorem. (An excellent introduction to the use of the Borsuk–Ulam theorem and related results in discrete geometry is Matoušek’s book [70].) Sarkaria [82] classified the  $d$ -complexes that are minimally non-embeddable into  $\mathbb{R}^{2d}$  (that is, every proper subcomplex is embeddable) under some additional technical assumptions and for  $d \neq 2$ , since only then is the Van Kampen obstruction complete. These complexes, the *Kuratowski complexes*, are simplicial complexes that are joins of  $(s-1)$ -skeletons of  $2s$ -simplices for possibly varying  $s \geq 1$ .

In the case of maps from a graph to the plane generically only two edges will intersect in a point. A new phenomenon for higher-dimensional complexes of low codimension in  $\mathbb{R}^d$  is multiple intersections of faces. Most generally, we will be interested in the intersection pattern of pairwise disjoint faces in a simplicial complex when mapped to a Euclidean space (or a manifold). For reasons that will become apparent very soon, we will refer to any such result as a result of *Tverberg-type*. These results present a vast and natural generalization of the theory of graph planarity and graph drawings. We have already pointed out that some care is needed in distinguishing results for affine and merely continuous maps. Most interestingly and perhaps surprisingly, divisibility properties of the number of faces involved in the intersection play a major role in determining which intersection patterns that must occur for affine maps also must occur for all continuous maps.



Of course, multiple intersections of faces already appear for maps of 2-dimensional complexes to the plane. Historically, this case is the starting point of Tverberg-type theory; see the survey by Ziegler [107]: Birch [11] showed that any  $3r$  points in  $\mathbb{R}^2$  can be partitioned into  $r$  intersecting triangles. If we care about optimality (we do!), we should note – as did Birch – that in fact any  $3r - 2$  points in the plane can be partitioned into  $r$  sets of points whose convex hulls intersect in a point. As for Radon’s theorem we can restate this as follows: any affine  $f: \Delta_{3(r-1)} \rightarrow \mathbb{R}^2$  identifies  $r$  points taken from pairwise disjoint faces of  $\Delta_{3(r-1)}$ . This led Birch to conjecture that any  $(r - 1)(d + 1) + 1$  points in  $\mathbb{R}^d$  can be partitioned into  $r$  sets with convex hulls intersecting in a common point. This conjecture was proven by Tverberg [97].

This raises the question whether the topological generalization of Birch’s or even Tverberg’s result to continuous maps remains true: given integers  $r \geq 2$  and  $d \geq 1$ , does every continuous map  $f: \Delta_{(r-1)(d+1)} \rightarrow \mathbb{R}^d$  identify points from  $r$  pairwise disjoint faces? That this should be true became known as the topological Tverberg conjecture. In 1976 Imre Bárány wrote a letter to Helge Tverberg asking about the topological version of his theorem, noting that there is a topological Radon theorem (due to Bajmóczy and Bárány), which is the case  $r = 2$  of the topological Tverberg conjecture. Tverberg then asked this question for a general  $(r - 1)(d + 1)$ -polytope in place of the simplex  $\Delta_{(r-1)(d+1)}$  in Oberwolfach in May 1978, see Gruber and Schneider [45]. (Thanks to Imre Bárány, Rolf Schneider, and Günter Ziegler for helpful remarks concerning the history of this question!)

The topological Tverberg conjecture as stated above was proven for  $r$  a prime by Bárány, Shlosman, and Szűcz [8]. Then the action of  $\mathbb{Z}/r$  by a shift of coordinates on  $\{(x_1, \dots, x_r) \in (\mathbb{R}^d)^r \mid \sum x_i = 0\}$  is free away from the origin. For  $r$  a power of a prime the symmetric group  $\mathfrak{S}_r$  has an elementary abelian subgroup that acts transitively and without fixed points on this vector space. Özaydin [75] was able to use this to prove the topological Tverberg conjecture for  $r$  a prime power. The topological Tverberg conjecture holds trivially for  $d = 1$ . (In fact, on the real line the image of a face under a continuous map contains the convex hull of the images of the vertices. Thus, the affine version of Tverberg’s theorem implies the continuous one for  $d = 1$ .)

Since Özaydin’s 1987 paper no progress had been made on deciding any further cases of the topological Tverberg conjecture. According to Matoušek [70, p. 154] it is one of the most challenging problems in topological combinatorics. We will construct counterexamples to the topological Tverberg conjecture for any  $r$  that is not a power of a prime. The construction builds on an  $r$ -fold version of the Van Kampen–Shapiro result on the embeddability of complexes in Euclidean space that was recently announced by Mabillard and Wagner [67]. For this they proved an  $r$ -fold version of the Whitney trick. The main obstacle to get from the result of Mabillard and Wagner to counterexamples of the topological Tverberg conjecture is

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that one has to circumvent the condition that the codimension of the complex in Euclidean space be at least three. Indeed, note that the dimension of the simplex  $\Delta_{(r-1)(d+1)}$  is much larger than  $d$ . Our main insight will be that a combinatorial reduction argument can eliminate this obstacle. This combinatorial reduction is the constraint method which was first introduced in joint work of Blagojević, Ziegler, and the author [16]. We will remark later on how surprisingly versatile and powerful this combinatorial trick really is.

Since there was no progress on the topological Tverberg conjecture for about 28 years, the focus of research had shifted away from only considering simplices to Tverberg-type problems in the full generality mentioned above. It is at this point that we come back to our goal of generalizing results about the planarity of graphs. While we have remarked on higher-dimensional versions of Kuratowski's theorem, we have thus far neglected to mention the corresponding multiple intersection results.

Sarkaria [81] proved an  $r$ -fold version of the Van Kampen–Flores theorem, that is, a Tverberg-type result for an appropriate skeleton of a simplex, for  $r$  prime. This was extended to prime power  $r$  by Volovikov [98]. This generalized Van Kampen–Flores theorem should be seen as the correct analogue of the non-planarity of  $K_5$ . As for the topological Tverberg conjecture the validity of this theorem depends on divisibility properties for the number  $r$  of faces: we will point out that the generalized Van Kampen–Shapiro theorem of Mabillard and Wagner implies that the generalized Van Kampen–Flores theorem of Sarkaria and Volovikov fails for every  $r$  that is not a prime power.

Tverberg-type results for complexes that have a partition into color classes such that no edge has vertices of the same color are due to Bárány and Larman [7] for dimension  $d = 2$  and Živaljević and Vrećica [109] for general  $d$ . These results and the generalization in Vrećica and Živaljević [99] specialize to the non-planarity of  $K_{3,3}$  for  $d = 2$  and  $r = 2$ . Moreover, Blagojević, Matschke, and Ziegler [18] were able to prove such a colored version for  $r \geq 3$  a prime and for the optimal  $N = (r - 1)(d + 1)$ .

The original proofs of all these Tverberg-type theorems require involved machinery from algebraic topology, where the case  $r$  prime is somewhat simpler than  $r$  a prime power. A key insight in this dissertation is that most of these results follow from the topological Tverberg theorem via the constraint method mentioned above. Since the original proofs of the Van Kampen–Flores and colored relatives of the topological Tverberg theorem are more involved than the proof of the original topological Tverberg theorem, this simplifies the proofs and presents a general and elementary framework to prove a large class of Tverberg-type results, many of them new or stronger than previously known versions.

The constraint method gives an effective way to restrict and control the combinatorics of a simplicial complex and obtain results of a topological nature, where continuous maps to Euclidean space are a benchmark for the complexity of the topology of the complex. One

could wish for a more direct approach to determining the influence of combinatorics on the topology of a complex. This will be developed for manifold triangulations in Chapter 3 of this dissertation. Restricting to manifold triangulations also presents a chance to investigate the interplay of combinatorics and geometry.

In dimension two, combinatorics, geometry, and topology are tightly connected via the Euler characteristic. Combinatorially, the Euler characteristic  $\chi(T)$  of a surface triangulation  $T$  of a surface  $M$  is  $\chi(T) = f_0 - f_1 + f_2$ , where  $(f_0, f_1, f_2)$  is the  $f$ -vector, that is,  $f_i$  counts the number of  $i$ -faces. This number  $\chi(T)$  is independent of the triangulation  $T$  and equal to  $\chi(M) = \text{rk } H_0(M; \mathbb{Q}) - \text{rk } H_1(M; \mathbb{Q}) + \text{rk } H_2(M; \mathbb{Q})$  the alternating sum of ranks of rational homology groups — a purely topological definition. Geometrically, the Euler characteristic can be defined (for a smooth surface) as  $\frac{1}{2\pi} \int_M \kappa \, dA$ , the integral over the Gauß curvature divided by  $2\pi$ .

The Gauß–Bonnet theorem asserts that these three definitions coincide. As a corollary we obtain that bounding local combinatorics of a surface triangulation leads to a global volume bound and a finite number of triangulations. We can explain this phenomenon in the following manner: start with a triangle and inductively build a triangulation. For this choose an arbitrary vertex in the boundary of the currently built part of the triangulation and close a disk of five triangles around this vertex. This process necessarily terminates. The largest triangulation we can build in this way is the boundary of the icosahedron. On the other hand by repeatedly gluing six triangles around a vertex we can tile the entire plane and the triangulation does not have to close up after a finite amount of time.

The difference between triangulations of degree five and six can be seen by a simple counting argument. By double-counting the  $f$ -vector of a surface triangulation is  $(f_0, 3f_0 - 3\chi(T), 2f_0 - 2\chi(T))$ . If every vertex has at most  $t$  incident edges, then  $2f_1 \leq tf_0$  and, thus,  $6f_0 - 6\chi(T) \leq tf_0$ . Equivalently,  $(6 - t)f_0 \leq 6\chi(T)$  and this remains true if  $t$  is the average vertex degree in  $T$ . In fact, for the average vertex degree  $t$  we get equality  $(6 - t)f_0 = 6\chi(T)$ . Since  $\chi(T) \leq 2$  for any surface triangulation,  $t \leq 5$  immediately implies  $f_0 \leq 12$ . The sign of  $6 - t$  is equal to the sign of  $\chi(T)$  and thus decides the geometry of the surface: spherical for  $t < 6$ , Euclidean for  $t = 6$ , and hyperbolic for  $t > 6$ .

In more geometric terms the average angle of a Euclidean triangle is  $\frac{\pi}{3}$ , and thus a disk of five triangles around a vertex has an angle defect of  $2\pi - 5\frac{\pi}{3} = \frac{\pi}{3}$  on average. Angle defects are a discrete analogue of curvature and the total curvature of a surface is bounded from above by  $4\pi$ . This again gives a bound of twelve vertices. Moreover, there is a spherical triangle with angles equal to  $\frac{2\pi}{5}$ . Inducing this metric on every facet of a triangulation with constant degree five yields a spherical metric on the underlying surface. Hence, the underlying surface is necessarily compact and we cannot build infinitely large triangulations with constant degree five.

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Generalizing the combinatorial reasoning to higher dimensions leads to global volume bounds from local volume bounds for manifold triangulations. The inequality  $f_1 \geq 3f_0 - 6$  can be generalized using the lower bound theorem, proven by Barnette [10] for polytopes, by Walkup [103] for manifold triangulations in dimensions three and four, by Kalai [53] for manifold triangulations in arbitrary dimension, and by Tay [93] for pseudomanifolds. The lower bound theorem asserts that the number of facets  $f_d$  of a  $d$ -dimensional manifold triangulation is bounded below by  $df_0 - (d+2)(d-1)$ . If we now have the local volume bound that on average every vertex is contained in fewer than  $d(d+1)$  facets, then for some  $\varepsilon > 0$  we have by the lower bound theorem

$$d(d+1) - \varepsilon \geq \frac{(d+1)f_d}{f_0} \geq \frac{(d+1)(df_0 - (d+2)(d-1))}{f_0} = d(d+1) - \frac{(d+2)(d+1)(d-1)}{f_0}.$$

This is equivalent to  $f_0 \leq \frac{(d+2)(d+1)(d-1)}{\varepsilon}$  and thus we have a global volume bound. The local bound is tight in the sense that there are infinitely large triangulations where every vertex is contained in precisely  $d(d+1)$  facets. This is related to results of Swartz [91] that for any bound  $c$  on  $g_2 = f_1 - (d+1)f_0 + (d+1)d$  there are only finitely many  $d$ -dimensional manifolds that admit a triangulation with  $g_2$  bounded by  $c$ .

Generalizing the geometric reasoning to higher dimensions we are interested in the possible dihedral angles of (regular) simplices in isotropic geometries. The dihedral angle of a  $d$ -simplex at a  $(d-2)$ -face (a *subridge*) is the opening angle of the two adjacent  $(d-1)$ -faces (*ridges*). In every dimension there is a regular spherical simplex with dihedral angles  $\frac{\pi}{2}$ . It can be obtained from radially projecting the crosspolytope outward onto the unit sphere. In low dimensions there are two additional regular simplicial polytopes larger than the crosspolytope. While in the crosspolytope every subridge is contained in precisely four facets – we will refer to that number as the *valence* of the subridge – there are simplicial 3- and 4-polytopes of constant valence five: the icosahedron and the 600-cell. Projecting these onto the unit sphere yields triangles, respectively tetrahedra, with dihedral angles equal to  $\frac{2\pi}{5}$ .

A regular Euclidean  $d$ -simplex has dihedral angles  $\arccos(\frac{1}{d})$ ; see Parks and Wills [77] for an elementary calculation. For  $d \leq 3$  the dihedral angle  $\arccos(\frac{1}{d})$  is less than  $\frac{2\pi}{5}$ , which explains the existence of constant valence five triangulations of  $S^2$  and  $S^3$ . Given any triangulation of a 3-manifold with constant valence five, inducing the metric of the regular spherical simplex from the 600-cell on every facet gives a homogeneous spherical metric on this manifold. Thus, it is a quotient of the 3-sphere and, in fact, a combinatorial quotient of the 600-cell.

This suggests that in any dimension  $d$  the triangulations with maximal valence four (or even maximal valence five for  $d = 2$  or  $d = 3$ ) are spherical. Moreover, there are finitely

many such triangulations in each dimension since we can obtain a volume bound from metric geometry. We will show that this is in fact true. Furthermore, we obtain a complete combinatorial classification of such triangulations. After explaining some interesting 3-dimensional examples we will generalize their constructions and thus obtain many symmetric triangulations of 3-manifolds of maximal valence six.

Brady, McCammond, and Meier [21] proved that any closed 3-manifold admits a triangulation with valences bounded by six. We will give a simple combinatorial proof of that fact and obtain a related stronger result that additionally bounds the number of combinatorially distinct vertex links by three. Previously, Cooper and Thurston [28] showed that there are five vertex links that suffice to triangulate any closed 3-manifold. Our improvement had already been observed by Walker (as pointed out in [28]) but was never published.

Interesting in regard to the influence of combinatorial restrictions on topological and geometric properties is Shephard's conjecture, that the boundary of any 3-polytope can be unfolded into the plane without overlap by cutting along edges (while remaining strongly connected). It is known that there are neither combinatorial nor metric obstructions to this conjecture. However, so far it has not been possible to combine these two statements into a proof of the conjecture.

Important steps towards a positive resolution of this conjecture were the insights that there are no metric obstructions to unfolding 3-polytopes via the *source unfolding* of Sharir and Schorr [85] and the *star unfolding* of Aronov and O'Rourke [3] and that there are no combinatorial obstructions, that is, every combinatorial 3-polytope is affinely equivalent to a polytope that has a net; see Ghomi [44].

Aronov and O'Rourke showed that the boundary of a 3-polytope can be unfolded in the plane if one allows cuts not only along edges but also across 2-faces. In their approach the metric space that can be isometrically embedded into the plane is obtained by cutting along shortest paths from a fixed vertex to all other vertices. The resulting isometric embedding is the star unfolding. This is essentially tied to the 3-dimensional case since Aronov and O'Rourke make use of a lemma of Alexandrov that every cellular 2-sphere with a facet-wise Euclidean metric and angles smaller than  $2\pi$  around vertices is realizable as the boundary of a convex polytope. This fails in higher dimensions.

The source unfolding of Sharir and Schorr unfolds a polytope starting from some point not on an edge along shortest paths, that is, via the exponential map at this point. The problem here is to show that this maps facets (locally) isometrically into the plane. Sharir and Schorr work around this by slicing the polytope open along the cut locus, the closure of sets of points with more than one shortest path to the source, and choosing a path for every (connected component of a) facet and unfolding entire facets along the way. To show that this gives a well-defined and injective unfolding they further cut the polytope boundary

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into peels, obtained by cutting along segments connecting the source to each vertex. This restricts Sharir and Schorr’s approach to dimension three.

More recently, Miller and Pak [73] gave a proof that there is no metric obstruction to unfolding any  $d$ -dimensional polyhedral space with angles of at most  $2\pi$  around any subridge into  $\mathbb{R}^d$ . They investigate the locus of points at a constant distance from a fixed point and define an unfolding via the exponential map, similar to Sharir and Schorr’s approach. Additionally, they are interested in questions of computational complexity.

Here we point out that a correct tool to understand metric obstructions that come from the combinatorics of the polytope are geometric structures induced by the face lattice. One can obtain a Euclidean structure on the boundary of a 3-polytope away from the vertices. Shepherd’s conjecture then can be investigated in terms of star-shaped regions in the universal covering of a polytope with vertices removed.

**A brief summary of the main contributions of this dissertation.** This dissertation provides a combinatorial toolbox to approach problems concerning the topology, geometry, and embeddability of cell complexes. We develop the constraint method for Tverberg-type results and give numerous applications of it, reproving and strengthening several Tverberg-type results via a combinatorial reduction from the topological Tverberg theorem. These results are joint work with Pavle Blagojević and Günter M. Ziegler. Most importantly, this gives an elementary route to circumvent the codimension requirement in work of Mabillard and Wagner, and thus gives rise to counterexamples to the topological Tverberg conjecture. We will attempt to give some indication of the different nature of affine and continuous maps from simplicial complexes to Euclidean space and how this depends on divisibility properties of the number of faces involved in an intersection. In particular, we will also show that the generalized Van Kampen–Flores result fails when the number of faces is not a prime power.

This combinatorial black box approach can be used in other settings, but we will not develop this theory in this dissertation. Rather, we will hint at the versatility of the constraint method by reproving a result of Dobbins on the barycenters of points in a certain polytope skeleton. This is joint work with Pavle Blagojević and Günter M. Ziegler.

The third chapter contains joint work with Frank Lutz and John M. Sullivan. We will present combinatorial constructions to obtain manifold triangulations of low valence, thus reproving a result of Brady, McCammond, and Meier [21] on 3-dimensional manifold triangulations with valence bounds in an elementary way. For higher dimensions we provide the first valence bound that does not depend on the dimension: every PL-manifold admits a triangulation with valences of subridges bounded by nine. We show that certain local combinatorics for 3-dimensional triangulations lead to manifolds with isotropic geometry,

namely, if every triangle has two edges of valence six and one edge of valence  $k$ , then the underlying manifold is spherical for  $k = 3$ , Euclidean for  $k = 4$ , and hyperbolic for  $k = 5$ . Every closed 3-manifold admits a triangulation with only these local combinatorics and varying  $k \in \{3, 4, 5\}$ . This is a higher-dimensional analogue of the fact that every surface admits a triangulation with vertex degrees five, six, and seven, where a surface with constant degree five is spherical, degree six leads to Euclidean surfaces, and triangulations with constant degree seven are hyperbolic.

Not only will we make use of combinatorial reasoning to obtain topological or geometric results, we will also investigate the converse direction. We will construct interesting triangulations of the 3-sphere via the Hopf fibration and employ metric geometry to deduce combinatorial volume bounds – and thus finiteness results – for triangulations with valence bounds.

In the last chapter we will give a conceptually simple proof of the result of Miller and Pak [73] that there are no metric obstruction to Shephard’s conjecture in any dimension. Miller and Pak use the exponential map and unfold along shortest paths. Instead of the exponential map defined on the boundary of the polytope itself, we construct unfoldings using the developing map of the Euclidean structure on the (universal covering of the) polytope boundary away from the subridges. This approach is very similar to Miller and Pak’s. The advantage of our method is that instead of developing along shortest paths we can develop along geodesics in the universal covering and thus we are not tied to Dirichlet domains. This allows for some flexibility.





## 2 The constraint method and Tverberg-type results

### Abstract

We reprove (and improve) several Tverberg-type results in an elementary fashion by deducing them from the topological Tverberg theorem via a combinatorial reduction: the constraint method. We also apply this method to prove that any point in an  $nd$ -polytope is the barycenter of  $n$  points in the  $d$ -faces, which was previously observed by Dobbins [32]. We circumvent the codimension requirement in the generalized Van Kampen–Shapiro–Wu theorem of Mabillard and Wagner [67] to show the existence of counterexamples to the topological Tverberg conjecture.

**Publication remark.** The results of Sections 2.2 through 2.8 are joint with Pavle Blagojević and Günter M. Ziegler and most of them can be found in [16], as are the results of Section 2.10 that appeared in the preprint [15]. The results of Section 2.9 mostly appeared in [42].

Tverberg-type results are concerned with the intersection pattern of faces in a simplicial complex when mapped to some Euclidean space (or manifold) by an affine or merely continuous map. We will be interested in the following problem: for given  $d$  and  $r$  characterize the simplicial complexes  $K$  such that for any continuous map  $f: K \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $K$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ . Such a collection of faces will be called an  *$r$ -Tverberg partition for  $f$* . Points  $x_i \in \sigma_i$  with  $f(x_1) = \dots = f(x_r)$  are called *points of Tverberg coincidence for  $f$* , while the intersection point  $f(x_i)$  is an  *$r$ -fold Tverberg point for  $f$* .

Complete answers to the problem above are not even known if we restrict to simplices only instead of general simplicial complexes. However, we will show that the characterization of simplices  $\Delta_N$  such that any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg points depends on divisibility properties of the number  $r$ , thus making an important step towards a more complete understanding of the problem. Moreover, we will be able to deduce numerous Tverberg-type result for general simplicial complexes from such results for simplices only. We provide a general combinatorial framework to generate such results in an elementary way,

## 2 The constraint method and Tverberg-type results

and thus reprove and strengthen most Tverberg-type results that were proven by involved techniques from algebraic topology in the past.

### 2.1 The topological Tverberg theorem

Tverberg determined the optimal dimension of a simplex  $\Delta_N$ , such that any affine map  $f: \Delta_N \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point.

**Theorem 2.1** (Tverberg [97]). *Let  $r \geq 2$  and  $d \geq 1$  be integers and  $N = (r - 1)(d + 1)$ . Then for any affine map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

This  $N$  is indeed optimal. An intersecting family of simplices in  $\mathbb{R}^d$  of dimensions  $d_1, \dots, d_r$  with  $d_i \leq d$  generically intersect in a set of codimension  $\sum_i d - d_i = rd - \sum_i d_i$ . Thus, we need  $d - rd + \sum_i d_i \geq 0$ , which is equivalent to  $\sum_i d_i + 1 \geq (r - 1)(d + 1) + 1$ . We can also explicitly construct a very non-generic map  $\Delta_{(r-1)(d+1)-1} \rightarrow \mathbb{R}^d$  without  $r$ -fold Tverberg point by splitting the vertices into  $d + 1$  parts of  $r - 1$  points and then sending each part to one of the vertices of  $\Delta_d \subseteq \mathbb{R}^d$ .

Generalizations of Theorem 2.1 to continuous maps have been of considerable interest. The Radon case  $r = 2$  was settled positively by Bajmóczy and Bárány [5] and led Tverberg to ask whether there are such versions of his more general theorem in Oberwolfach in May 1978. Tverberg formulated his question even for any polytope of dimension  $N$  in place of the simplex, since this is the version that was proven for  $r = 2$ . Notice though, that if every continuous map  $f: \Delta_{(r-1)(d+1)} \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point then so does every continuous map  $\partial P \rightarrow \mathbb{R}^d$  from the boundary of an  $(r - 1)(d + 1)$ -polytope to  $\mathbb{R}^d$ , since polytopes are PL and thus are subdivisions of the simplex; see Grünbaum [48, Sec. 11.1] or [46] for this specific statement. Tverberg's question was quickly settled for  $r$  a prime:

**Theorem 2.2** (Bárány, Shlosman, and Szűcz [8]). *Let  $r \geq 2$  be a prime,  $d \geq 1$  be an integer, and  $N = (r - 1)(d + 1)$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

It is worthwhile to present the outline of two possible proof strategies of this theorem. The first is similar to the proof given in [8], but reformulated in the way it can be found in Matoušek's book [70]. The second proof outlined below is due to Vučić and Živaljević [101]. Their setup can be used to count the number of Tverberg partitions. Some improvements in that direction are due to Hell [50].

At the heart of both proof strategies is the configuration space – test map scheme. We will refer to [70] for any notation and facts. Here we briefly repeat the most important facts:

## 2.1 The topological Tverberg theorem

By  $\text{pt}$  we denote the one-point space and  $[r]$  denotes the discrete space with  $r$  points. The *join* of two abstract (vertex-disjoint) simplicial complexes  $K$  and  $L$  is the simplicial complex  $K * L = \{\sigma \cup \tau \mid \sigma \in K, \tau \in L\}$ . Its geometric realization is the join of the geometric realizations of  $K$  and  $L$  as topological spaces. Given topological spaces  $X$  and  $Y$  we can represent points in the join  $X * Y$  abstractly as convex combinations  $\lambda x + \mu y$  with  $\lambda, \mu \geq 0$ ,  $\lambda + \mu = 1$ , and  $x \in X$ ,  $y \in Y$ . Here we have to take care whenever one of the coefficients  $\lambda$  or  $\mu$  vanishes, since  $0x + 1y = 0x' + 1y$  for all  $x, x' \in X$  and  $y \in Y$ . The *pairwise deleted join* of  $K$  with itself is defined by  $K_{\Delta(2)}^{*2} = \{\sigma * \tau \mid \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$ . We will denote by  $K^{*n}$  and by  $K_{\Delta(2)}^{*n}$  the  $n$ -fold join of  $K$  and the  $n$ -fold pairwise deleted join of  $K$ , respectively. Joins and pairwise deleted joins of simplicial complexes can be interchanged:  $(K^{*n})_{\Delta(2)}^{*r} \cong (K_{\Delta(2)}^{*r})^{*n}$ .

Let  $N = (r - 1)(d + 1)$  for a prime  $r \geq 2$  and an integer  $d \geq 1$ . Let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be an arbitrary continuous map. Consider the configuration space  $(\Delta_N)_{\Delta(2)}^{*r}$  of joins of  $r$ -tuples of pairwise disjoint faces in  $\Delta_N$ . This simplicial complex is isomorphic to

$$(\Delta_N)_{\Delta(2)}^{*r} \cong (\text{pt}^{*(N+1)})_{\Delta(2)}^{*r} \cong (\text{pt}_{\Delta(2)}^{*r})^{*(N+1)} \cong [r]^{*(N+1)}.$$

By the general fact  $\text{conn } X * Y \geq \text{conn } X + \text{conn } Y + 2$  for the connectivity of the join and  $\text{conn}[r] = -1$  we have  $\text{conn}(\Delta_N)_{\Delta(2)}^{*r} = N - 1$ .

Now we can define the test map

$$\Phi: (\Delta_N)_{\Delta(2)}^{*r} \rightarrow (\mathbb{R}^{d+1})^r, \lambda_1 x_1 + \cdots + \lambda_r x_r \mapsto (\lambda_1, \lambda_1 f(x_1), \dots, \lambda_r, \lambda_r f(x_r)).$$

This map is  $\mathfrak{S}_r$ -equivariant, where the symmetric group  $\mathfrak{S}_r$  acts by permuting copies of  $\Delta_N$  on the domain and by permuting copies of  $\mathbb{R}^{d+1}$  on the codomain. Let  $D = \{(y_1, \dots, y_r) \in (\mathbb{R}^{d+1})^r \mid y_1 = \cdots = y_r\}$  be the diagonal. Then  $\Phi(\lambda_1 x_1 + \cdots + \lambda_r x_r) \in D$  implies that  $\lambda_1 = \cdots = \lambda_r = \frac{1}{r}$  and thus  $f(x_1) = \cdots = f(x_r)$ . The quotient map  $\widehat{\Phi}: (\Delta_N)_{\Delta(2)}^{*r} \rightarrow (\mathbb{R}^{d+1})^r / D$  induced by  $\Phi$  maps to 0 if and only if  $f$  has an  $r$ -fold Tverberg point. The diagonal  $D$  is exactly the set of points in  $(\mathbb{R}^{d+1})^r$  that have non-trivial stabilizers under the action of the cyclic subgroup  $\mathbb{Z}/r$  that acts by shifting copies. Here we use that  $r$  is prime. Thus,  $\mathbb{Z}/r$  acts freely on  $(\mathbb{R}^{d+1})^r / D$  away from the origin. Observe that  $\dim(\mathbb{R}^{d+1})^r / D = (r - 1)(d + 1) = N = \text{conn}(\Delta_N)_{\Delta(2)}^{*r} + 1$ . Now, we finish the proof by applying a lemma of Dold to show that  $\widehat{\Phi}$  has a zero.

**Lemma 2.3** (Dold [33], see also Matoušek [70]). *Let a non-trivial finite group  $G$  act on an  $n$ -connected CW complex  $T$  and act linearly on an  $(n + 1)$ -dimensional real vector space  $V$ . Suppose that the action of  $G$  on  $V \setminus \{0\}$  is free. Then any  $G$ -equivariant map  $\widehat{\Phi}: T \rightarrow V$  has a zero.*

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We can also show the existence of a zero of the map  $\widehat{\Phi}$  by degree theoretic methods, similar to the proof by Vučić and Živaljević [101]. For this we first need to find a suitable orientable manifold of dimension  $N - 1$  inside  $(\Delta_N)_{\Delta(2)}^{*r}$ . To this end we first observe that  $S^1$  embeds into  $(\Delta_1)_{\Delta(2)}^{*r}$  even in a  $\mathbb{Z}/r$ -equivariant way, with  $\mathbb{Z}/r$  acting freely on the circle  $S^1$ . Let us suppose that  $r \neq 2$ . Then in particular  $r$  is odd and  $N = (r - 1)(d + 1)$  is even. By taking joins we obtain a  $\mathbb{Z}/r$ -equivariant embedding  $S^{N-1} \cong (S^1)^{*N/2} \rightarrow ((\Delta_1)_{\Delta(2)}^{*r})^{*N/2}$ . By interchanging the order of the join and deleted join the codomain is isomorphic to  $(\Delta_{N-1})_{\Delta(2)}^{*r}$ . If  $\widehat{\Phi}$  does not have a zero, we obtain a  $\mathfrak{S}_r$ -map  $(\Delta_N)_{\Delta(2)}^{*r} \rightarrow S^{N-1}$  by normalizing. Thus, we have the composition of maps  $S^{N-1} \rightarrow (\Delta_{N-1})_{\Delta(2)}^{*r} \rightarrow (\Delta_N)_{\Delta(2)}^{*r} \rightarrow S^{N-1}$  that we will denote by  $\Psi$ . Since  $\Psi$  factors through the  $(N - 1)$ -connected space  $(\Delta_N)_{\Delta(2)}^{*r}$  we know that  $\deg \Psi = 0$ . However, the identity is a  $\mathbb{Z}/r$ -equivariant map  $S^{N-1} \rightarrow S^{N-1}$  of degree one, and any equivariant map between free (closed and orientable)  $G$ -manifolds has the same degree modulo the order of the group; see Kushkuley and Balanov [61]. This is a contradiction.

It might seem reasonable to expect that Theorem 2.2 can be extended to an arbitrary number  $r$  of faces. In 1987 Özaydin was able to prove such an extension for  $r$  a power of a prime with methods from equivariant cohomology theory.

**Theorem 2.4** (Topological Tverberg theorem: Özaydin [75]). *Let  $r \geq 2$  be a prime power,  $d \geq 1$  be an integer, and  $N = (r - 1)(d + 1)$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

The cases for  $r$  having at least two distinct prime divisors have remained open since then. We will build on work of Mabillard and Wagner [67] to construct counterexamples in Section 2.9, thus disproving the following conjecture:

**Conjecture 2.5** (Topological Tverberg conjecture). *Let  $r \geq 2$  and  $d \geq 1$  be integers, and  $N = (r - 1)(d + 1)$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

## 2.2 A general principle

It was generally believed that to prove Tverberg-type results for more general complexes than simplices one has to extend the topological methods and results of the previous section. However, one can use the topological Tverberg theorem as a black box to derive other results by a combinatorial reduction argument based on the pigeonhole principle.

**Lemma 2.6.** *Let  $r \geq 2$  be a prime power, let  $d \geq 1$  and  $c \geq 1$  be integers, and  $N = (r - 1)(d + c + 1)$ . Then given any continuous maps  $f: \Delta_N \rightarrow \mathbb{R}^d$  and  $g: \Delta_n \rightarrow \mathbb{R}^c$  we find  $r$  points of Tverberg coincidence for  $f$ ,  $f(x_1) = \dots = f(x_r)$  with  $g(x_1) = \dots = g(x_r)$ .*

## 2.2 A general principle

*Proof.* Consider the map  $\Delta_N \rightarrow \mathbb{R}^{d+c}, x \mapsto (f(x), g(x))$  and use the topological Tverberg theorem 2.4. □

We will use the map  $g$  to constrain Tverberg partitions. Consequently, we refer to  $g$  as a *constraint function*. The same theorem holds true for any  $r$  if  $N$  is replaced by the least integer  $M$  such that every continuous map  $\Delta_M \rightarrow \mathbb{R}^{d+c}$  admits an  $r$ -fold Tverberg point. Upper and lower bounds on this  $M$  will be investigated in Section 2.9.

**Theorem 2.7.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$  be an integer, and  $N = (r-1)(d+2)$ . Let  $\Sigma$  be a simplicial complex on more than  $N$  vertices such that for any partition of the vertex set of  $\Sigma$  into  $r$  parts  $\sigma_1, \dots, \sigma_r$  at least one part is a face of  $\Sigma$ . Then for any continuous map  $f: \Sigma \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* We can think of  $\Sigma$  as a subcomplex of the simplex  $\Delta_M$ , where  $M+1 > N$  is the number of vertices of  $\Sigma$ . Extend the map  $f$  in an arbitrary way to a map  $F: \Delta_M \rightarrow \mathbb{R}^d$ . Consider the constraint function  $g: \Delta_M \rightarrow \mathbb{R}, x \mapsto \text{dist}(x, \Sigma)$ , where  $\text{dist}$  is any continuous function that is zero precisely on  $\Sigma$ , for example, the distance to  $\Sigma$  in a suitable metric on  $\Delta_N$ . The map  $g$  is continuous since  $\Sigma$  is a closed subset of  $\Delta_M$ . By Lemma 2.6 there exist  $r$  points of Tverberg coincidence for  $F$ ,  $F(x_1) = \dots = F(x_r)$  that all have the same distance to  $\Sigma$ . Let  $\sigma_i$  be the support of  $x_i$ , that is, the minimal face of  $\Delta_M$  containing  $x_i$ . Then the  $\sigma_i$  are pairwise disjoint and contained in a partition of the vertex set of  $\Sigma$ . Thus, one  $\sigma_j$  is a face of  $\Sigma$  and  $g(x_j) = 0$ . Then  $g(x_i) = 0$  for all  $i = 1, \dots, r$ . This implies  $\sigma_i \subseteq \Sigma$  for all  $i$ , since the  $x_i$  are in the relative interior of the  $\sigma_i$ . □

This suggests a more general approach: Given a continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  and integers  $r \geq 2$  and  $d \geq 1$ , call a subcomplex  $\Sigma \subseteq \Delta_N$  *Tverberg unavoidable for  $f$*  if for every Tverberg partition  $\sigma_1, \dots, \sigma_r$  for  $f$  at least one  $\sigma_i$  is a face of  $\Sigma$ . Sometimes we will abbreviate “Tverberg unavoidable for  $f$ ” to just “unavoidable.” Clearly the proof of Theorem 2.7 works for any Tverberg unavoidable subcomplex  $\Sigma \subseteq \Delta_N$  for  $f$ . Moreover, we can also iterate this to force the Tverberg partition into an intersection of Tverberg unavoidable complexes for larger  $N$ .

**Theorem 2.8.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$  and  $c \geq 1$  be integers, and  $N = (r-1)(d+c+1)$ . Let a continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  be given and let  $\Sigma_1, \dots, \Sigma_c \subseteq \Delta_N$  be Tverberg unavoidable for  $f$ . Then there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  in  $\Sigma_1 \cap \dots \cap \Sigma_c$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* The proof is essentially the same as for Theorem 2.7, but now we consider the constraint function  $g: \Delta_N \rightarrow \mathbb{R}^c, x \mapsto (\text{dist}(x, \Sigma_1), \dots, \text{dist}(x, \Sigma_c))$ . □

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In light of the counterexamples to the topological Tverberg conjecture in Section 2.9 we really need to require that  $r$  is a power of a prime. However, for an arbitrary integer  $r$  we get a version of Theorem 2.8, where  $N = (r - 1)(d + c + 1)$  has to be replaced by the least integer  $N$  such that every continuous map  $\Delta_N \rightarrow \mathbb{R}^{d+c}$  has an  $r$ -fold Tverberg point. The corollaries we will derive from Theorem 2.8 below hold in a similar way whenever  $r$  is not a power of a prime. In fact, this is a key insight into constructing counterexamples to the topological Tverberg conjecture.

For  $N = (r - 1)(d + c + 1)$  we do not need to require that  $r$  is a prime power if the map  $f$  as well as the constraint function  $g$  are affine. In this case we can use Tverberg's original theorem as a black box result. For  $\Sigma \subseteq \Delta_N$  the map  $g: \Delta_N \rightarrow \mathbb{R}, x \mapsto \text{dist}(x, \Sigma)$  is affine if and only if  $\Sigma$  is an induced subcomplex, that is,  $\Sigma$  is a subsimplex of  $\Delta_N$ . We can use this to show that the (topological) Tverberg theorem for dimension  $d + 1$  and  $r$  faces implies the validity of the theorem for dimension  $d$  and the same number of faces. Also see de Longueville [63, Prop. 2.5] for this observation, where an extended target space and the unavoidability of a certain subcomplex is used.

**Theorem 2.9.** *Let  $r \geq 2$ ,  $d \geq 1$ , and  $N \geq r - 1$  be integers. If every affine (respectively every continuous) map  $F: \Delta_N \rightarrow \mathbb{R}^{d+1}$  admits an  $r$ -fold Tverberg point, then every affine (respectively every continuous) map  $f: \Delta_{N-r+1} \rightarrow \mathbb{R}^d$  admits an  $r$ -fold Tverberg point.*

*Proof.* We can think of  $\Delta_{N-r+1}$  as a subcomplex of  $\Delta_N$ . As remarked above  $\Delta_N \rightarrow \mathbb{R}, x \mapsto \text{dist}(x, \Delta_{N-r+1})$  is an affine map. It remains to be shown that  $\Delta_{N-r+1}$  is Tverberg unavoidable. Let  $\sigma_1, \dots, \sigma_r$  be faces of  $\Delta_N$  that are pairwise disjoint. Then at most  $r - 1$  of them can have a vertex outside of  $\Delta_{N-r+1}$ . Thus, one face  $\sigma_j$  is contained in  $\Delta_{N-r+1}$ .  $\square$

## 2.3 Weak colored versions of the topological Tverberg theorem

While investigating the number of halving planes of a point set in  $\mathbb{R}^3$  Bárány, Füredi, and Lovász [6] realized the need for a “colored version” of Tverberg's theorem, at least for points in the plane. They prove the following:

**Lemma 2.10** (Bárány, Füredi, and Lovász [6, Lemma 3]). *There is a positive integer  $t$ , such that given  $3t$  points in general position in  $\mathbb{R}^2$  partitioned into three sets  $A, B$  and  $C$ , each of cardinality  $t$ , there exist nine points  $a_i \in A$ ,  $b_i \in B$ ,  $c_i \in C$ ,  $i = 1, 2, 3$ , with  $\bigcap_i \text{conv}\{a_i, b_i, c_i\} \neq \emptyset$ .*

Bárány, Füredi, and Lovász point out that they can prove that  $t \leq 4$  but for brevity's sake only present a proof of  $t \leq 7$ . Phrasing Lemma 2.10 in terms of affine maps  $f: \Delta_{3t-1} \rightarrow \mathbb{R}^2$

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we think of  $A$ ,  $B$ , and  $C$  as colors assigned to the vertices of  $\Delta_{3t-1}$ . Faces that do not contain edges with both endpoints in the same color class are called *rainbow*. The subcomplex  $R$  of  $\Delta_{3t-1}$  of all rainbow faces is the *rainbow complex*. Lemma 2.10 is then a Tverberg-type result for the rainbow complex corresponding to the colors  $A$ ,  $B$ , and  $C$  and  $r = 3$ .

As  $r = 3$  is a prime power we can try to apply our general setup of unavoidable subcomplexes, that is, attempt to obtain Lemma 2.10 as a corollary of the topological Tverberg theorem. Let  $t \leq 5$ , then the subcomplex of faces of  $\Delta_{2t-1}$  that have at most one vertex in  $A$  is unavoidable. In fact, let us note the following more general lemma for a single color class  $A$ .

**Lemma 2.11.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$ , and  $N = (r - 1)(d + 2)$ . Let  $A \subseteq \Delta_N$  be any set of at most  $2r - 1$  vertices, and let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be any continuous map. Then there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $|\sigma_i \cap A| \leq 1$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* The subcomplex  $\Sigma$  of faces with at most one vertex in  $A$  is Tverberg unavoidable: let  $\sigma_1, \dots, \sigma_r$  be pairwise disjoint faces of  $\Delta_N$  and suppose that each  $\sigma_i$  has at least two vertices in  $A$ . Then all the  $\sigma_i$  have at least  $2r$  vertices in  $A$  since they are vertex-disjoint, but this is in contradiction to  $|A| \leq 2r - 1$ . Thus,  $\Sigma$  contains a Tverberg partition by Theorem 2.8.  $\square$

If we assume that  $|A| \geq r$  then we can also state this theorem with equalities  $|\sigma_i \cap A| = 1$  by adding a point in  $A$  to every face  $\sigma_i$  that is disjoint from  $A$ . The following “colored Radon theorem” is a direct consequence for  $r = 2$ .

**Corollary 2.12** (Colored Radon: Vrećica and Živaljević [100, Cor. 7]). *Let  $d \geq 1$ , and let the map  $f: \Delta_{d+2} \rightarrow \mathbb{R}^d$  be continuous and let  $A \subseteq \Delta_{d+2}$  be a set of three vertices. Then there are disjoint faces  $\sigma_1, \sigma_2$  in  $\Delta_{d+2}$  with  $|\sigma_1 \cap A| \leq 1$  and  $|\sigma_2 \cap A| \leq 1$  such that  $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$ .*

Coming back to Lemma 2.10 we have shown that the rainbow complex is an intersection of three unavoidable complexes, one for each of the three color classes. Thus, by Theorem 2.8 the rainbow complex has a 3-fold Tverberg point if  $N \geq (r - 1)(d + c + 1) = 2 \cdot 6 = 12$ . This means we obtain a Tverberg-type result for  $t = 5$ , since then  $3t - 1 \geq N = 12$ . Of course, we can additionally forget about two arbitrary vertices.

Bárány and Larman [7] proved sharper versions and generalized to any  $r$  and  $d = 2$ . Let the vertices of  $\Delta_{3r-1}$  be partitioned into three color classes of cardinality  $r$ . Denote the corresponding rainbow complex by  $R$ . The main result of [7] is that any affine map  $f: R \rightarrow \mathbb{R}^2$  has an  $r$ -fold Tverberg point. Bárány and Larman asked whether this generalizes to higher dimensions. In particular, they asked for the least integer  $N = N(r, d)$  such that if the vertices of  $\Delta_{N-1}$  are partitioned into  $d + 1$  color classes of cardinality at least  $r$ , then

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any affine map  $f: R \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point. Here  $R \subseteq \Delta_{N-1}$  denotes again the corresponding rainbow complex. Bárány and Larman conjectured that the trivial lower bound  $N(r, d) \geq r(d+1)$  is tight, and their result above shows that this is true for  $d = 2$ .

We will refer to the statement  $N(r, d) = r(d+1)$  as the *Bárány–Larman conjecture*. Still in the same paper [7] an argument of Lovász is presented showing  $N(2, d) = 2(d+1)$  using the Borsuk–Ulam theorem. Indeed, for  $r = 2$  the rainbow complex of  $d+1$  color classes of size two is obtained from  $\Delta_{2(d+1)-1}$  by deleting a perfect matching from the 1-skeleton. This yields the boundary of the  $(d+1)$ -dimensional crosspolytope. An application of the Borsuk–Ulam theorem finishes the proof of  $N(2, d) = 2(d+1)$  noticing that for any  $x$  with support  $\sigma$  in the crosspolytope the antipodal point  $-x$  is contained in the disjoint face  $-\sigma$ .

This suggests several routes to explore: application of the Borsuk–Ulam theorem does not require an affine map. Thus, the proof above works to show that  $N(r, d) = 2(d+1)$  even for the more general topological analogue of the Bárány–Larman conjecture. As with the general Tverberg theorem it is interesting to explore whether there might be differences between the affine and continuous versions of this conjecture.

Moreover, for the case  $r = 2$  the points of Tverberg coincidence are not arbitrary points within their disjoint faces but they are precisely antipodal points of each other. Another way to phrase this is that both points  $x$  and  $-x$  have the same barycentric coordinates with respect to each color class. That is, if  $x = \sum \alpha_i v_i$  is a convex combination of vertices  $v_i$ , where  $v_i$  has color  $i$ , then  $-x = \sum \alpha_i (-v_i)$  has the same coefficients in the convex combination, where the vertex  $-v_i$  again has color  $i$ . Soberón [87] generalized this result for affine maps to arbitrary  $r$  and that will be the topic of Section 2.5. There we will also give a simple proof of Soberón’s result using our constraint setup and in the same way obtain a continuous analogue of his theorem for  $r$  a prime power.

Lastly, even establishing that the numbers  $N(r, d)$  are finite (exist) for every  $r$  and  $d$  is a highly non-trivial problem. Blagojević, Matschke, and Ziegler [18] proved the Bárány–Larman conjecture for  $r+1$  a prime. Their approach can also be used to show that  $N(r, d) \leq 2(d+1)(r-1) + 1$  for  $r$  prime [13]. Other bounds on  $N(r, d)$  are not known. We will remark on their theorem and related results in Section 2.4.

Coming back to the problem that motivated these colored versions of Tverberg’s theorem – finding asymptotic upper bounds on the number of halving planes of a point set in  $\mathbb{R}^3$  – one actually does not need to bound  $N(r, d)$  to obtain higher-dimensional versions, that is, bounds on the number of halving hyperplanes for point sets in  $\mathbb{R}^d$ . For this it is sufficient to establish that the numbers  $T(r, d)$  are finite, where  $t = T(r, d)$  is the least integer such that every affine  $f: R \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point, where  $R$  is the rainbow complex of  $d+1$  color classes of size at least  $t$ . The problem was introduced by Živaljević and Vrećica [109]. They established the finiteness of  $T(r, d)$  for any  $r$  and  $d$  by topological means. In particular,



### 2.3 Weak colored versions of the topological Tverberg theorem

their results still hold true for merely continuous  $f: R \rightarrow \mathbb{R}^d$ .

**Theorem 2.13** (Živaljević and Vrećica [109]). *For any prime  $r$  and  $d \geq 1$  an integer, we have  $T(r, d) \leq 2r - 1$ . This implies  $T(r, d) \leq 4r - 1$  for arbitrary  $r$ .*

The bound  $T(r, d) \leq 2r - 1$  can be extended to prime powers  $r$ , as noted by Živaljević in [108]. Our proof of Lemma 2.10 was not tied to the fact that  $d = 2$ , so we can immediately generalize it to higher dimensions  $d$ . We thus obtain a result that is slightly stronger than Theorem 2.13.

**Theorem 2.14** (Weak colored Tverberg). *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $N = (r - 1)(2d + 2)$  and let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be continuous. If the vertices of  $\Delta_N$  are colored by  $d + 1$  colors, where each color class has cardinality at most  $2r - 1$ , then there are  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* For each fixed color  $i$ , the subcomplex  $\Sigma_i$  of faces that have at most one vertex of color  $i$  is Tverberg unavoidable.  $\square$

Note that in Theorem 2.14 the fact that all vertices can be colored with  $d + 1$  colors of size at most  $2r - 1$  implies that  $N + 1 \leq (2r - 1)(d + 1)$ . The theorem is “weak” as it needs a large number of points/vertices to reach its conclusion, namely  $N + 1 = (r - 1)(2d + 2) + 1$ , while the optimal result requires only  $N + 1 = (r - 1)(d + 1) + 1$  of them, as in Theorem 2.16 below. The special case of Theorem 2.14 when all color classes have the same cardinality  $2r - 1$ , and thus  $N + 1 = (d + 1)(2r - 1)$ , is Theorem 2.13. As we do *not* need to require all color classes to have the same size (a simple observation that apparently was first used by Blagojević, Matschke, and Ziegler [18]), we need  $d$  points/vertices less to force a colored Tverberg partition.

Theorem 2.14 leaves some flexibility in the choice of color classes: For example, we could consider  $d$  colors of cardinality  $2r - 2$  and one color class of size  $2r - 1$ . Instead of shrinking the size of color classes we can also allow fewer color classes. This gives a colored Tverberg theorem “of type B” (in terminology of Vrećica and Živaljević introduced in [99]), that is, fewer than  $d + 1$  colors are possible. This naturally leads to Tverberg-type theorems, where the dimensions of faces in a Tverberg partition are bounded. We will discuss these theorems in Section 2.6.

We obtain the following strengthening of Theorem 2.14, which in the special case where all color classes have the same size is the main result of [99].

**Theorem 2.15.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $c \geq \lceil \frac{r-1}{r}d \rceil + 1$ , and  $N = (r - 1)(d + 1 + c)$ . Let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be continuous. If the vertices of  $\Delta_N$  are divided into  $c$  color classes, each of them of cardinality at most  $2r - 1$ , then there are  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

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*Proof.* We need that all  $N + 1$  vertices can be colored by  $c$  colors of cardinality at most  $2r - 1$  to use Theorem 2.8, that is, we need  $c(2r - 1) \geq N + 1$ , which is equivalent to  $c \geq \lceil \frac{r-1}{r}d \rceil + 1$ .  $\square$

## 2.4 Optimal colored versions of the topological Tverberg theorem

To apply the constraint method of Theorem 2.8 we need more than the optimal number  $(r-1)(d+1)+1$  of points. However, even for the optimal number, that is, for the topological Tverberg theorem, one can restrict the Tverberg partitions. In other words there is a proper subcomplex  $\Sigma \subseteq \Delta_{(r-1)(d+1)}$  such that each continuous  $f: \Sigma \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point, at least for the case  $r \geq 3$ .

**Theorem 2.16** (Blagojević, Matschke, and Ziegler [18]). *Let  $r \geq 2$  be a prime,  $d \geq 1$  be an integer, and  $N = (r-1)(d+1)$ . Suppose the  $N + 1$  vertices of  $\Delta_N$  are partitioned into color classes  $C_0, \dots, C_m$  each of size at most  $r - 1$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

As corollaries Blagojević, Matschke, and Ziegler obtain that  $T(r, d) = r$  for  $r+1$  a prime, which implies  $T(r, d) \leq 2r-2$  for arbitrary  $r$  and the asymptotic bound  $T(r, d) \leq (1+o(1))r$  for  $r \rightarrow \infty$ .

For Theorem 2.8 we used the topological Tverberg theorem as a black box result to reduce other Tverberg-type results to it via a purely combinatorial reduction. Theorem 2.16 is properly stronger than the topological Tverberg theorem for  $r$  an odd prime and so we can use it as a black box result in place of the topological Tverberg theorem. We will give some applications of this in this section.

**Theorem 2.17.** *Let  $r \geq 2$  be a prime,  $d \geq 1$ ,  $\ell \geq 0$ , and  $k \geq 0$ . Let the vertices of  $\Delta_N$  be colored by  $\ell+k$  colors  $C_0, \dots, C_{\ell+k-1}$  with  $|C_0| \leq r-1, \dots, |C_{\ell-1}| \leq r-1$  and  $|C_\ell| \geq 2r-1, \dots, |C_{\ell+k-1}| \geq 2r-1$ , where  $|C_0| + \dots + |C_{\ell-1}| > (r-1)(d-k+1) - k$ . Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* Without loss of generality we can assume that  $|C_\ell| = \dots = |C_{\ell+k-1}| = 2r-1$ , by deleting any additional vertices. Then the simplex  $\Delta_N$  has still  $N + 1 = |C_0| + \dots + |C_{\ell-1}| + k(2r-1)$  vertices, so

$$N = |C_0| + \dots + |C_{\ell-1}| + k(2r-1) - 1 \geq (r-1)(d+k+1).$$

## 2.5 Colored versions with equal barycentric coordinates

Now we split each of the color classes  $C_\ell, \dots, C_{\ell+k-1}$  into new color sub-classes of cardinality at most  $r - 1$ . (For example, singletons will do.) Let  $\Sigma_i$  be the subcomplex of all faces of  $\Delta_N$  with at most one vertex in  $C_i$ . Thus Theorem 2.16 together with the proof technique of Theorem 2.8 yields that there is a Tverberg  $r$ -partition  $\sigma_1, \dots, \sigma_r$  where each of the simplices  $\sigma_i$  is a rainbow simplex with respect to the refined coloring where the large color classes have been split into sub-classes, and it also lies in  $\Sigma_\ell \cap \dots \cap \Sigma_{\ell+k-1}$ , that is, it uses at most one of the color sub-classes of each of  $C_\ell, \dots, C_{\ell+k-1}$  and thus respects the original coloring.  $\square$

This Theorem 2.17 contains Theorem 2.16 as the special case  $k = 0$ , and also Vrećica and Živaljević's [100, Prop. 5] as the special case  $|C_0| = \dots = |C_{\ell-1}| = r - 1$  and  $|C_\ell| = \dots = |C_{\ell+k-1}| = 2r - 1$ . If we further specialize to  $r = 2$  and  $k = 1$ , this in turn reduces to the “colored Radon” Corollary 2.12, as noted in [100, Cor. 7]. For  $\ell = 0$  we get Vrećica and Živaljević's colored Tverberg theorem “of type B,” see [99] and [100, Cor. 8].

As a second instance of combining Theorem 2.16 with the proof technique of Theorem 2.8, we obtain from our method the following new result about colored Tverberg partitions with restricted dimensions. For this we need the unavoidability of the  $\lceil \frac{r-1}{r}d \rceil$ -skeleton of  $\Delta_{(r-1)(d+2)}$ :

**Lemma 2.18.** *Let  $r \geq 2$  and  $d \geq 1$  be integers. Let  $N = (r - 1)(d + 2)$  and  $k \geq \lceil \frac{r-1}{r}d \rceil$ . Then the  $k$ -skeleton  $\Delta_N^{(k)}$  of  $\Delta_N$  is Tverberg unavoidable.*

*Proof.* Let  $\sigma_1, \dots, \sigma_r$  be a partition of the vertex set of  $\Delta_N$  into  $r$  parts, and suppose that  $|\sigma_i| \geq k + 2$  for all  $i$ . Then the  $\sigma_i$  involve at least  $r(k + 2) \geq (r - 1)d + 2r > N + 1$  vertices. This is a contradiction since the  $\sigma_i$  can involve at most  $N + 1$  vertices. Thus, there is a  $j$  such that  $|\sigma_j| \leq k + 1$ , which means  $\sigma_j$  is a face of  $\Delta_N^{(k)}$ .  $\square$

As an immediate consequence of this and using Theorem 2.16 as a black box (in place of the topological Tverberg theorem as in Theorem 2.8) we get the following:

**Theorem 2.19.** *Let  $r \geq 2$  be a prime,  $d \geq 1$ ,  $N = (r - 1)(d + 2)$ , and  $k \geq \lceil \frac{r-1}{r}d \rceil$ . Let the vertices of  $\Delta_N$  be colored by  $m + 1$  colors  $C_0, \dots, C_m$  with  $|C_i| \leq r - 1$  for all  $i$ . Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $\dim \sigma_i \leq k$  for  $1 \leq i \leq r$ , such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

## 2.5 Colored versions with equal barycentric coordinates

Let the vertices of  $\Delta_N$  be partitioned into  $\ell$  color classes  $C_0, \dots, C_{\ell-1}$ . Every point  $x \in R$  in the corresponding rainbow complex  $R$  has unique barycentric coordinates  $x = \sum_{i=0}^{\ell-1} \alpha_i v_i$  with  $0 \leq \alpha_i \leq 1$  and  $v_i$  a vertex in the color class  $C_i$  for  $0 \leq i \leq \ell - 1$ . We say that two points  $x, y$  in the rainbow complex have *equal barycentric coordinates* if  $x = \sum_{i=0}^{\ell-1} \alpha_i v_i$

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and  $y = \sum_{i=0}^{\ell-1} \alpha_i w_i$ , where  $v_i$  and  $w_i$  are vertices in the color class  $C_i$ . The following theorem is a topological version of Soberón's [87] "Tverberg's theorem with equal barycentric coordinates."

**Theorem 2.20** (Topological Tverberg with equal barycentric coordinates). *Let  $r \geq 2$  be a prime power,  $d \geq 1$ , and  $N = r((r-1)d+1) - 1$ . Let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be continuous. If the vertices of  $\Delta_N$  are partitioned into  $(r-1)d+1$  color classes of size  $r$ , then there are points  $x_1, \dots, x_r$  with equal barycentric coordinates in  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  whose images intersect, with  $f(x_1) = \dots = f(x_r)$ .*

*Proof.* Let the color classes be  $C_0, \dots, C_{(r-1)d}$ . Every point  $x \in \Delta_N$  is a unique convex combination  $x = \sum \alpha_i v_i$  of the vertices of  $\Delta_N$ . For  $0 \leq k \leq (r-1)d$  let  $g_k: \Delta_N \rightarrow \mathbb{R}$  be given by  $\sum \alpha_i v_i \mapsto \sum_{v_i \in C_k} \alpha_i$ . Each  $g_k$  is an affine function that is equal to 1 on the simplex  $\text{conv}(C_k) \subseteq \Delta_N$  with vertex set  $C_k$  and 0 on all other vertices of  $\Delta_N$ .

By Lemma 2.6 there are  $x_1, \dots, x_r \in \Delta_N$  with  $x_i \in \sigma_i$ , where the  $\sigma_i \subseteq \Delta_N$  are pairwise disjoint and  $f(x_1) = \dots = f(x_r)$  as well as  $g_k(x_1) = \dots = g_k(x_r)$  for  $1 \leq k \leq (r-1)d$ ; that is, the lemma does not guarantee equality for  $g_0$ . However, as  $g_0 + \dots + g_{(r-1)d} = 1$  we also obtain  $g_0(x_1) = \dots = g_0(x_r)$ .

Suppose that for some  $k$ , the face  $\sigma_j$  has at least one vertex in  $C_k$ . As we may again assume that  $\sigma_j$  is the minimal face of  $\Delta_N$  that contains  $x_j$ , this implies that  $g_k(x_j) \neq 0$  and hence  $g_k(x_i) \neq 0$  for  $1 \leq i \leq r$ . Thus, all  $r$  faces  $\sigma_i$  have at least one vertex in  $C_k$ . However, as  $|C_k| = r$  and the  $\sigma_i$  are pairwise disjoint, every  $\sigma_i$  has exactly one vertex in  $C_k$ . Since this is true for every color, the  $\sigma_i$  belong to the rainbow complex.

Thus, the numbers  $g_k(x_i)$  for  $0 \leq k \leq (r-1)d$  are exactly the barycentric coordinates of  $x_i$ . These are equal for all the  $x_i$  since  $g_k(x_1) = \dots = g_k(x_r)$  for all  $k$ .  $\square$

In the preceding theorem the number  $N$  is large enough to allow for  $(r-1)d$  constraints. Think of the simplex  $\Delta_{(r-1)d}$  as embedded into  $\mathbb{R}^{(r-1)d}$ . Then the constraint function could be defined as the affine function mapping all vertices in color class  $C_i$  to vertex number  $i$  of  $\Delta_{(r-1)d}$ .

Soberón in [87, Section 4] suggests an alternative idea for how to obtain the topological analogue of his result that we have obtained using our ansatz. We also obtain Soberón's original result in the same way:

**Theorem 2.21** (Tverberg with equal barycentric coordinates: Soberón [87]). *Let  $r \geq 2$ ,  $d \geq 1$ , and  $N = r((r-1)d+1) - 1$ . Let  $f: \Delta_N \rightarrow \mathbb{R}^d$  be affine. If the vertices of  $\Delta_N$  are partitioned into  $(r-1)d+1$  color classes of size  $r$ , then there are points  $x_1, \dots, x_r$  with equal barycentric coordinates in  $r$  pairwise disjoint rainbow faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  whose images intersect, with  $f(x_1) = \dots = f(x_r)$ .*

*Proof.* The proof is the same as for Theorem 2.20. The constraint functions are affine.  $\square$

Soberón shows that the number of color classes and the number of points per color class are optimal for the theorem to hold. Thus, the topological version, Theorem 2.20, is also optimal in that sense.

## 2.6 Prescribing dimensions for faces in a Tverberg partition

Tverberg’s theorem concerns maps from a high-dimensional complex to Euclidean space of lower dimension. Oftentimes one is interested in Tverberg-type theorems for complexes of positive codimension in the target space. For example, such a result will be a key step towards the counterexamples of Section 2.9. Of particular interest are results about embeddings of complexes. In fact, one such non-embeddability result predates Tverberg’s theorem by more than three decades.

**Theorem 2.22** (Van Kampen–Flores theorem: Van Kampen [54] and Flores [38]). *For  $d \geq 1$  and every continuous map  $f: \Delta_{2d+2}^{(d)} \rightarrow \mathbb{R}^{2d}$  there are two disjoint faces  $\sigma_1$  and  $\sigma_2$  of  $\Delta_{2d+2}^{(d)}$  with  $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$ .*

Here  $K^{(d)}$  denotes the  $d$ -skeleton of  $K$ , that is, the faces in  $K$  of dimension at most  $d$ . This theorem follows quite simply from Theorem 2.7: given a partition of the  $2d+3$  vertices of  $\Delta_{2d+2}$  into two sets, one of them must have at most  $d+1$  elements. Before generalizing this to  $r$ -fold intersections, we will make some remarks.

For  $d = 1$  the Van Kampen–Flores theorem establishes the non-planarity of  $K_5$  in the strong form that even two vertex-disjoint edges intersect under  $f$ . A simple reformulation of Theorem 2.22 is that any map  $f: \Delta_{d+2}^{(\lceil d/2 \rceil)} \rightarrow \mathbb{R}^d$  has a 2-fold Tverberg point, where the odd-dimensional case follows by including  $\mathbb{R}^d$  into  $\mathbb{R}^{d+1}$ . We will give a strengthening of this theorem below.

Comparing the Van Kampen–Flores theorem to the topological Radon theorem, that is, the  $r = 2$  case of the topological Tverberg theorem that any map  $f: \Delta_{d+1} \rightarrow \mathbb{R}^d$  has a 2-fold Tverberg point, we see that adding a point to  $\Delta_{d+1}$  allows us to bound the dimension of faces in a Radon partition.

As we have just seen the topological Radon theorem implies the Van Kampen–Flores theorem. However, chronologically the Van Kampen–Flores theorem (1932/33) was proven half a century earlier than the topological Radon theorem (1979). In fact, the two theorems can be deduced from one another: Let  $f: \Delta_{d+1} \rightarrow \mathbb{R}^d$  be continuous. Then we can regard the join of the map  $f$  with itself as a map to  $\mathbb{R}^{2d+1}$ , explicitly, consider  $F: \Delta_{d+1} * \Delta_{d+1} \rightarrow$

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$\mathbb{R}^{2d+1}$ ,  $\lambda_1 x_1 + \lambda_2 x_2 \mapsto (\lambda_1, \lambda_1 f(x_1), \lambda_2 f(x_2))$ . Now  $F(\lambda_1 x_1 + \lambda_2 x_2) = F(\mu_1 y_1 + \mu_2 y_2)$  implies  $\lambda_1 = \mu_1$  and, thus,  $f(x_1) = f(y_1)$  or for  $\lambda_1 = 0$  we have  $f(x_2) = f(y_2)$ . This shows that if  $F$  has a 2-fold Tverberg point then so does  $f$ , and  $F$  has such a point of coincidence by the Van Kampen–Flores theorem since  $\Delta_{d+1} * \Delta_{d+1} \cong \Delta_{2d+3}$ . We will use this construction in Section 2.9 to investigate the asymptotics of the optimal  $N$  for maps  $\Delta_N \rightarrow \mathbb{R}^d$  to have an  $r$ -fold Tverberg point if  $r$  is not a power of a prime.

**Theorem 2.23** (Sarkaria [81] and Volovikov [98]). *Let  $r \geq 2$  be a prime power,  $d \geq 1$  and  $k \geq \frac{r-1}{r}d$  be integers, and  $N = (r-1)(d+2)$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$  and  $\dim \sigma_i \leq k$  for all  $i = 1, \dots, r$ .*

This  $N$  is optimal as can be seen by placing  $r-1$  points at each vertex of the  $d$ -simplex in  $\mathbb{R}^d$  and  $r-1$  points in the barycenter. The statements of Sarkaria’s and Volovikov’s theorems are much more involved and more general than the theorem we stated above. We will discuss the prerequisites of these theorems in Section 2.8, where we discuss  $j$ -wise disjoint Tverberg-type theorems.

*Proof of Theorem 2.23.* This is a corollary of Theorem 2.7 and Lemma 2.18, which states that the  $k$ -skeleton of  $\Delta_N$  is Tverberg unavoidable.  $\square$

**Example 2.24.** For  $d = r = 3$  and  $f$  an affine map, this theorem asserts that given eleven points in  $\mathbb{R}^3$ , one can find three pairwise disjoint sets of three points whose convex hulls intersect. Ten points are not sufficient for this, as by the discussion above one needs more than  $(r-1)(d+2) = 10$  points. (This solves a problem discussed by Matoušek in [70, Example 6.7.4].)

It is noteworthy that no affine version of Theorem 2.23 is known for  $r$  not a prime power. Not even any bound on  $N$  is known in these cases. Moreover, some cases of Theorem 2.23 are wrong for  $r$  not a power of a prime and for any  $N$ , see Section 2.9. A proof of an affine version of the Van Kampen–Flores theorem via Gale transforms is due to Soberón [87, Corollary 8].

We can interpolate between colored versions of the topological Tverberg theorem and versions with bounded dimension of the faces in a Tverberg partition.

**Theorem 2.25.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$  and  $c \geq 0$  be integers, and  $N = (r-1)(d+c+1)$ . Let  $A_1, \dots, A_c$  pairwise disjoint sets of vertices of  $\Delta_N$  and let  $a_1, \dots, a_c$  be non-negative integers with  $|A_i| \leq (a_i+1)r-1$  for each  $i$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$  and each  $\sigma_i$  has at most  $a_j$  vertices in  $A_j$ .*

## 2.6 Prescribing dimensions for faces in a Tverberg partition

*Proof.* We only need to check that the complex of faces with at most  $a_j$  vertices in the set  $A_j$  is Tverberg unavoidable to use Theorem 2.8. But this is again an immediate consequence of the pigeonhole principle: if each face  $\sigma_i$  has at least  $a_j+1$  vertices in  $A_j$ , then  $|A_j| \geq r(a_j+1)$ , a contradiction. Thus, the subcomplex of faces with at most  $a_j$  vertices in each  $A_j$  is the intersection of  $c$  Tverberg unavoidable subcomplexes.  $\square$

This theorem has the following special cases:

- For  $c = 0$  this gives the topological Tverberg theorem (but, of course, we used this theorem in the proof).
- For  $c = 1$  we get Theorem 2.23.
- For  $a_1 = \dots = a_c = 1$  we get Theorems 2.14 and 2.15.
- More generally, for  $a_1 = \dots = a_c$  this theorem was recently reproven by Jojić, Vrećica, and Živaljević [51] via shellability of multiple chessboard complexes.
- For  $r = 2$  and  $|A_i| = 2a_i + 1$  we get the non-embeddability of Sarkaria's Kuratowski complexes [82].

One can ask for a further strengthening of Theorem 2.23 where we would not put the same dimension bound on all simplices  $\sigma_i$ .

**Theorem 2.26.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $N \geq (r-1)(d+2)$ , and*

$$r(k+1) + s > N + 1 \quad \text{for integers } k \geq 0 \text{ and } 0 \leq s < r.$$

*then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ ,  $\dim(\sigma_i) \leq k$  for all  $i$ , and the number  $\ell$  of simplices  $\sigma_i$  with  $\dim \sigma_i = k$  satisfies  $\ell(k+1) \leq N - (r-s) + 1$ .*

*Proof.* The complex  $\Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)}$  is Tverberg unavoidable: If none of the faces  $\sigma_1, \dots, \sigma_r$  lies in  $\Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)}$ , then they all have dimension at least  $k$ , and since they are disjoint only  $r-s$  of them can involve one of the last  $r-s$  vertices, so  $s$  of them must have dimension at least  $k+1$ . For this  $r(k+1) + s$  vertices are needed.

Thus, there is a Tverberg partition  $\sigma_1, \dots, \sigma_r$  with all simplices in the unavoidable subcomplex, so we get  $\dim \sigma_i \leq k$ . Moreover, the simplices  $\sigma_i$  altogether can take up only  $N + 1 \leq r(k+1) + s$  vertices.  $\square$

The following generalization was conjectured in [16]:

**Conjecture 2.27.** *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $N \geq (r-1)(d+2)$ , and*

$$r(k+1) + s > N + 1 \quad \text{for integers } k \geq 0 \text{ and } 0 \leq s < r.$$

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then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ , with  $\dim \sigma_i \leq k + 1$  for  $1 \leq i \leq s$  and  $\dim \sigma_i \leq k$  for  $s < i \leq r$ .

This conjecture was recently proven by Jojić, Vrećica, and Živaljević [52] with topological methods and (as in [51]) via shellability of multiple chessboard complexes. They show the following:

**Theorem 2.28** (Jojić, Vrećica, and Živaljević [52]). *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $N \geq (r - 1)(d + 2)$ , and  $rk + s \geq (r - 1)d$  for integers  $k \geq 0$  and  $0 \leq s < r$ . Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$ , there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ , with  $\dim \sigma_i \leq k + 1$  for  $1 \leq i \leq s$  and  $\dim \sigma_i \leq k$  for  $s < i \leq r$ .*

More generally, Roland Bacher has asked on `mathoverflow` [4] which dimensions  $d_i = \dim \sigma_i$  could be prescribed for a Tverberg partition if the number of points  $N$  is sufficiently large. We have already noted that a Tverberg  $r$ -partition in which the codimensions of the  $\sigma_i$  add to more than  $d$  will not exist for an affine general position map, so we need to assume that  $\sum_i (d - d_i) \leq d$ . Also arbitrarily large families of  $N$  points on the moment curve, whose convex hulls are neighborly polytopes, show that we cannot force that  $\dim \sigma_i < \lfloor \frac{d}{2} \rfloor$  for any  $i$ . See also Haase [49].

**Theorem 2.29** (Van Kampen–Flores, sharpened). *Let  $d \geq 1$ . Then for every continuous map  $f: \Delta_{d+2} \rightarrow \mathbb{R}^d$  there are two disjoint faces  $\sigma_1, \sigma_2$  of  $\Delta_{d+2}$  such that  $\dim \sigma_1 = \lceil \frac{d}{2} \rceil$ ,  $\dim \sigma_2 = \lfloor \frac{d}{2} \rfloor$ , and  $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$ .*

*Proof.* The case  $d$  even is Theorem 2.22. It remains to settle the case when  $d$  is odd, with  $\dim \sigma_1 = \frac{d-1}{2}$  and  $\dim \sigma_2 = \frac{d+1}{2}$ . In terms of Theorem 2.26, in this situation we have  $r = 2$ ,  $k = \frac{d+1}{2}$ ,  $s = 1$ , and thus there is a Tverberg 2-partition  $\sigma_1, \sigma_2$  with  $\dim \sigma_i \leq \frac{d+1}{2}$ , where at most  $\ell \leq \lfloor \frac{N-(r-s)+1}{k+1} \rfloor = \lfloor \frac{(d+2)-(2-1)+1}{\frac{d+1}{2}+1} \rfloor = 1$  of the  $\sigma_i$  have dimension  $k = \frac{d+1}{2}$ .  $\square$

## 2.7 Embedding graphs into 3-space

Theorem 2.29 has consequences for embeddings of graphs into  $\mathbb{R}^3$ . Two disjoint closed curves in  $\mathbb{R}^3$  are *unlinked* if each can be filled with a disk that is disjoint from the other curve. Otherwise the closed curves are called *linked*. A graph  $G$  is called *intrinsically linked* if for every embedding  $f: G \rightarrow \mathbb{R}^3$  we can find two vertex-disjoint cycles in  $G$  whose images under  $f$  are linked curves. Here we will give a short and simple proof of the Conway–Gordon–Sachs theorem that  $K_6$  is intrinsically linked.

**Theorem 2.30** (Conway and Gordon [26], Sachs [80]). *The complete graph on six vertices  $K_6$  is intrinsically linked.*



*Proof.* Suppose there is an embedding  $f: K_6 \rightarrow \mathbb{R}^3$  such that any two vertex-disjoint cycles are unlinked. These cycles are necessarily 3-cycles. By definition of unlinked we can fill in these cycles with disks which are disjoint from the cycle supported on the other three vertices. In this way we can construct a map from the 2-skeleton  $\Delta_5^{(2)}$  of the 5-simplex to  $\mathbb{R}^3$ , where any 2-face is disjoint from any vertex-disjoint edge. This is in contradiction to Theorem 2.29.  $\square$

Actually this proof shows more: a 2-dimensional complex  $K$  admits a map  $f: K \rightarrow \mathbb{R}^3$  such that for  $\sigma_1$  a triangle in  $K$  and  $\sigma_2$  an edge vertex-disjoint from  $\sigma_1$  we have  $f(\sigma_1) \cap f(\sigma_2) = \emptyset$  if and only if the 1-skeleton  $K^{(1)}$  of  $K$  admits a linkless embedding into  $\mathbb{R}^3$ .

As another instance of a Hanani–Tutte theorem in dimension  $d = 3$  Robertson, Seymour, and Thomas [79] showed that a graph admits a linkless embedding if and only if it admits a *panelled* embedding. Here panelled means that the disks that show that curves are unlinked can be chosen disjoint from the rest of the graph, that is, also disjoint from edges that have vertices in common with the cycle but do not belong to the cycle itself.

Robertson, Seymour, and Thomas classify intrinsically linked graphs as those that have as a minor one of the seven graphs obtainable from  $K_6$  by  $Y - \Delta$  and  $\Delta - Y$  exchanges. This also classifies those complexes  $K$  such that for any map  $f: K \rightarrow \mathbb{R}^3$  there is a triangle  $\sigma_1$  and a disjoint edge  $\sigma_2$  with  $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$  as those whose 1-skeleton  $K^{(1)}$  have a minor as above.

We note that we could use Theorem 2.29 to obtain higher-dimensional versions of the Conway–Gordon–Sachs theorem.

## 2.8 $j$ -wise disjoint Tverberg partitions

We described Tverberg-type results as theorems about the intersection pattern of pairwise disjoint faces in a simplicial complex when mapped to a Euclidean space. We motivated the requirement of faces being pairwise disjoint by higher-dimensional versions of the Hanani–Tutte theorem. Here we will further reinforce that it is sufficient to investigate collections of pairwise disjoint faces by showing that  $j$ -wise disjoint versions of Tverberg-type result follow by another simple combinatorial reduction. A family of (not necessarily distinct) faces  $\sigma_1, \dots, \sigma_r$  is called  *$j$ -wise disjoint* if all subfamilies of at most  $j$  faces have empty intersection.

A  $j$ -wise disjoint version of the topological Tverberg theorem is due to Sarkaria for the number of faces  $r$  a prime, and due to Volovikov for  $r$  a prime power. We will give a simple proof below. Also, our proof extends to the affine case for any  $r$ , which is a new result.

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**Theorem 2.31** (*j*-wise disjoint topological Tverberg: Sarkaria [82], Volovikov [98]). *Let  $r \geq 2$  be a prime power,  $d \geq 1$ ,  $2 \leq j \leq r$ , and*

$$N + 1 > \frac{r-1}{j-1}(d + 1). \quad (2.1)$$

*Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $j$ -wise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* Let  $N' := (N + 1)(j - 1) - 1$ , let  $p$  be the natural simplicial projection  $\Delta_{N'} \cong \Delta_N^{*(j-1)} \rightarrow \Delta_N$  that maps each of the  $j - 1$  copies of a vertex  $v \in \Delta_N$  in the join  $\Delta_N^{*(j-1)}$  to the vertex  $v$ , and set  $f' := f \circ p: \Delta_{N'} \rightarrow \mathbb{R}^d$ .

We have  $N' \geq (r - 1)(d + 1)$ . Thus, by the topological Tverberg theorem, Theorem 2.4, there are pairwise disjoint faces  $\sigma'_1, \dots, \sigma'_r \subseteq \Delta_{N'}$  such that  $f'(\sigma'_1) \cap \dots \cap f'(\sigma'_r) \neq \emptyset$ . By definition of  $f'$  this is equivalent to  $f(p(\sigma'_1)) \cap \dots \cap f(p(\sigma'_r)) \neq \emptyset$ . Now, let  $\sigma_1 = p(\sigma'_1), \dots, \sigma_r = p(\sigma'_r)$ . The faces  $\sigma_i$  of  $\Delta_N$  are  $j$ -wise disjoint: if  $j$  of the faces  $\sigma_i$  had a vertex in common, then at least two of the faces  $\sigma'_i$  would share a vertex. However, these faces are pairwise disjoint.  $\square$

**Theorem 2.32** (*j*-wise disjoint Tverberg). *Let  $r \geq 2$ ,  $d \geq 1$ ,  $2 \leq j \leq r$ , and*

$$N + 1 > \frac{r-1}{j-1}(d + 1). \quad (2.2)$$

*Then for every affine map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $j$ -wise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* For affine maps  $f: \Delta_N \rightarrow \mathbb{R}^d$  we need not even assume that  $r$  is a prime power—if we use Tverberg’s original theorem as the black box result. This is possible since the projection map  $p: \Delta_N^{*(j-1)} \rightarrow \Delta_N$  as in the proof of Theorem 2.31 is affine.  $\square$

Sarkaria and Volovikov proved a  $j$ -wise disjoint version of Theorem 2.23. We will discuss the theorem below and then observe that we obtain a stronger result using our methods.

**Theorem 2.33** (Generalized Van Kampen–Flores: Sarkaria [81] and Volovikov [98]). *Let  $r \geq 2$  be a prime power,  $2 \leq j \leq r$ ,  $d \geq 1$ , and  $k < d$  such that there is an integer  $m \geq 0$  that satisfies*

$$(r - 1)(m + 1) + r(k + 1) \geq (N + 1)(j - 1) > (r - 1)(m + d + 2). \quad (2.3)$$

*Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$   $j$ -wise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $\dim \sigma_i \leq k$  for  $1 \leq i \leq r$ , such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

## 2.8 $j$ -wise disjoint Tverberg partitions

Let us discuss which of the conditions posed by (2.3) are necessary. First, the left-hand side has to be strictly larger than the right-hand side, which yields the condition  $(m+1)(r-1) + r(k+1) - (m+d+2)(r-1) > 0$ , that is,

$$k \geq \frac{r-1}{r}d. \quad (2.4)$$

This lower bound on  $k$  is necessary, as we see by looking at a generic affine map  $f$ , which does not have the desired Tverberg  $r$ -partition unless the sum of the codimensions of the  $\sigma_i$  is at most  $d$ , that is,  $r(d-k) \leq d$ .

If the second inequality in (2.3) is satisfied for some  $m \geq 0$ , then it is in particular satisfied for  $m = 0$ , which gives  $(N+1)(j-1) > (r-1)(d+2)$ , that is,

$$N+1 > \frac{r-1}{j-1}(d+2). \quad (2.5)$$

It is not clear whether this lower bound on  $N$  is necessary in general;

$$N+1 > \lfloor \frac{r-1}{j-1} \rfloor (d+2)$$

is necessary for  $k < d$ , as one can see from an affine map  $\Delta_N \rightarrow \Delta_d$  that maps at most  $\lfloor \frac{r-1}{j-1} \rfloor$  vertices of  $\Delta_N$  to each of the vertices and to the barycenter of a  $d$ -simplex. This example is suggested by Sarkaria in [81]. Note that for  $j = 2$  the lower bound of (2.5) reads  $N+1 > (r-1)(d+2)$ , which is optimal, despite a mistaken claim in [81, Thm. 1.5 and the sentence after this] that the bound  $N \geq r(s+1) - 2$  is optimal, where  $s = k+1$  in Sarkaria's notation. In the example he gives, the two bounds coincide.

Even if both conditions (2.4) and (2.5) hold the integer  $m$  that should satisfy (2.3) may not exist. This requirement is non-trivial, see Example 2.35 below. It is not necessary, as we shall see.

We now obtain our strengthening of Theorem 2.33 as a corollary of Theorem 2.23, the special case  $j = 2$ .

**Theorem 2.34** (Generalized Van Kampen–Flores, sharpened). *Let  $r \geq 2$  be a prime power,  $2 \leq j \leq r$ ,  $d \geq 1$ , and  $k \leq N$  such that*

$$k \geq \frac{r-1}{r}d \quad \text{and} \quad N+1 > \frac{r-1}{j-1}(d+2). \quad (2.6)$$

*Then for every continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$   $j$ -wise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$ , with  $\dim \sigma_i \leq k$  for  $1 \leq i \leq r$ , such that  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

The proof is the same as for Theorem 2.31 by observing that the natural projection  $p: \Delta_N^{*(j-1)} \rightarrow \Delta_N$  does not increase the dimension of faces.

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**Example 2.35.** To see that Theorem 2.34 is stronger than Theorem 2.33, let  $d = j = r = 3$  and  $k = 2$ . Then the prerequisites of Theorem 2.34 are satisfied for  $N = 5$ . Thus for any continuous map  $f: \Delta_5 \rightarrow \mathbb{R}^3$  there are three 3-wise disjoint faces of dimension at most 2 whose images intersect.

However, inequality (2.3) of Theorem 2.33 asks that  $(m + 1) \cdot 2 + 3 \cdot 3 \geq (N + 1) \cdot 2 > (m + 5) \cdot 2$ , that is,  $2m + 11 \geq 2N + 2 > 2m + 10$ . Such an integer  $m$  exists for no  $N$ , as  $2N + 2$  is even.

One can easily prove  $j$ -wise disjoint versions of the topological Tverberg theorem and its relatives without the requirement that  $r$  be a prime power.

**Theorem 2.36.** *Let  $r \geq 2$ ,  $d \geq 1$ , and  $2 \leq j \leq r$  be integers. Suppose  $r = (j - 1)q$ , where  $q \geq 2$  is a prime power and  $N = (q - 1)(d + 1)$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$   $j$ -wise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .*

*Proof.* There are  $q$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_q$  of  $\Delta_N$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ . Now take  $(j - 1)$  copies of each  $\sigma_i$ .  $\square$

**Example 2.37.** In particular, Theorem 2.31 is not tight. Let  $r = 4$ ,  $d = 2$ , and  $j = 3$ . Then Theorem 2.31 says that for any continuous map  $f: \Delta_4 \rightarrow \mathbb{R}^2$  there are four 3-wise disjoint simplices  $\sigma_1, \dots, \sigma_4$  of  $\Delta_4$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_4) \neq \emptyset$ . The same statement is true for  $f: \Delta_3 \rightarrow \mathbb{R}^2$  by Theorem 2.36 for  $q = 2$ , since every continuous map  $\Delta_3 \rightarrow \mathbb{R}^2$  has a 2-fold Tverberg point.

## 2.9 Counterexamples to the topological Tverberg conjecture

In this section we will show how a result of Mabillard and Wagner [67] implies counterexamples to the  $r$ -fold version of the generalized Van Kampen–Flores theorem for any  $r$  that is not a prime power. Since this result would be a corollary of the topological Tverberg conjecture, this also implies that this conjecture is wrong for any  $r$  that is not a power of a prime. For those cases we will investigate some techniques that can give insights (at least asymptotically) into the smallest  $N$  such that any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point.

For a simplicial complex  $K$  denote by

$$K_{\Delta(2)}^{\times r} = \{(x_1, \dots, x_r) \in \sigma_1 \times \dots \times \sigma_r \mid \sigma_i \text{ face of } K, \sigma_i \cap \sigma_j = \emptyset \forall i \neq j\}$$

the *pairwise deleted  $r$ -fold product* of  $K$ . The space  $K_{\Delta(2)}^{\times r}$  is a polytopal cell complex

## 2.9 Counterexamples to the topological Tverberg conjecture

in a natural way (its faces are products of simplices). Denote by  $W_r$  the vector space  $\{(x_1, \dots, x_r) \in \mathbb{R}^r \mid \sum x_i = 0\}$  with the action by the symmetric group  $\mathfrak{S}_r$  that permutes coordinates.

**Theorem 2.38** (Mabillard and Wagner [67, Theorem 3]). *Suppose that  $r \geq 2$ ,  $k \geq 3$ , and let  $K$  be a simplicial complex of dimension  $(r - 1)k$ . Then the following statements are equivalent:*

- (i) *There exists an  $\mathfrak{S}_r$ -equivariant map  $K_{\Delta(2)}^{\times r} \rightarrow S(W_r^{\oplus rk})$ .*
- (ii) *There exists a continuous map  $f: K \rightarrow \mathbb{R}^{rk}$  such that for any  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $K$  we have  $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$ .*

Thus, under certain requirements it is sufficient to construct an equivariant map from the configuration space  $K_{\Delta(2)}^{\times r}$  to some sphere in order to show the existence of a continuous map  $f: K \rightarrow \mathbb{R}^d$  without  $r$ -fold Tverberg point.

**Lemma 2.39.** *Let  $r \geq 6$  be an integer that is not a prime power, and let  $k \geq 3$ ,  $N \geq 1$  be integers. Let  $K = \Delta_N^{\binom{r-1}{k}}$ . Then there exists an  $\mathfrak{S}_r$ -equivariant map  $K_{\Delta(2)}^{\times r} \rightarrow S(W_r^{\oplus rk})$ .*

*Proof.* The configuration space  $K_{\Delta(2)}^{\times r}$  has dimension at most  $r(r - 1)k$ . Özaydin shows that there is an  $\mathfrak{S}_r$ -equivariant map from a finite, free,  $n$ -dimensional,  $(n - 1)$ -connected  $\mathfrak{S}_r$ -complex into  $S(W_r^{\oplus d})$ , where  $n = (r - 1)d$  [75, Theorem 4.2]. This also gives an equivariant map  $K_{\Delta(2)}^{\times r} \rightarrow S(W_r^{\oplus rk})$  for any  $n$ -dimensional, finite, and free  $\mathfrak{S}_r$ -complex by observing that such a complex always admits an equivariant map into an  $(n - 1)$ -connected  $\mathfrak{S}_r$ -space; see for example Matoušek [70, Lemma 6.2.2].  $\square$

To get a rough understanding how  $r$  being a prime power is related to the existence of equivariant maps, we review the lemma Özaydin proves to construct such maps. This will allow us to give an alternative proof of Lemma 2.39.

**Lemma 2.40** (Özaydin [75, Lemma 4.1]). *Let  $d \geq 3$  and  $G$  be a finite group. Let  $X$  be a  $d$ -dimensional free  $G$ -CW complex and let  $Y$  be a  $(d - 2)$ -connected  $G$ -CW complex. There is a  $G$ -map  $X \rightarrow Y$  if and only if there are  $G_p$ -maps  $X \rightarrow Y$  for every Sylow  $p$ -subgroup  $G_p$ ,  $p$  prime.*

Again, the existence of an  $\mathfrak{S}_r$ -equivariant map  $K_{\Delta(2)}^{\times r} \rightarrow S(W_r^{\oplus rk})$  is a simple consequence. The reasoning is the same as in [75, Proof of Theorem 4.2]: the free  $\mathfrak{S}_r$ -space  $K_{\Delta(2)}^{\times r}$  has dimension at most  $d = r(r - 1)k$ , and  $S(W_r^{\oplus rk}) \cong S^{(r-1)rk-1}$  is  $(d - 2)$ -connected. By Lemma 2.40 the existence of an  $\mathfrak{S}_r$ -map  $K_{\Delta(2)}^{\times r} \rightarrow S(W_r^{\oplus rk})$  reduces to the existence of equivariant maps for Sylow  $p$ -subgroups, but  $p$ -groups have fixed points in  $S(W_r^{\oplus rk})$  for  $r$  not a prime power by [75, Lemma 2.1], so a constant map will do.

Combining Lemma 2.39 with Theorem 2.38 gives the following corollary.

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**Corollary 2.41.** *Let  $r \geq 6$  be an integer that is not a prime power, and let  $k \geq 3$  be an integer. Then for any  $N$  there exists a continuous map  $f: \Delta_N \rightarrow \mathbb{R}^{rk}$  such that for any  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $\dim \sigma_i \leq (r-1)k$  we have  $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$ .*

We have already mentioned that if the topological Tverberg conjecture holds for some  $r$ , then the generalized Van Kampen–Flores result holds for the same  $r$ . Since Corollary 2.41 contradicts the  $r$ -fold Van Kampen–Flores theorem for  $r$  not a power of a prime, the topological Tverberg conjecture must fail for those  $r$ .

**Theorem 2.42** (The topological Tverberg conjecture fails). *Let  $r \geq 6$  be an integer that is not a prime power, and let  $k \geq 3$  be an integer. Let  $N = (r-1)(rk+2)$ . Then there exists a continuous map  $F: \Delta_N \rightarrow \mathbb{R}^{rk+1}$  such that for any  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  we have  $F(\sigma_1) \cap \dots \cap F(\sigma_r) = \emptyset$ .*

*Proof.* We will explicitly construct the map  $F$  from the map  $f$  whose existence is guaranteed by Corollary 2.41. Let  $f: \Delta_N \rightarrow \mathbb{R}^{rk}$  be a continuous map as constructed in Corollary 2.41, that is, such that for any  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  with  $\dim \sigma_i \leq (r-1)k$  we have  $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$ . Define  $F: \Delta_N \rightarrow \mathbb{R}^{rk+1}$ ,  $x \mapsto (f(x), \text{dist}(x, \Delta_N^{((r-1)k)}))$ . Suppose there were  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_N$  such that there are points  $x_i \in \sigma_i$  with  $F(x_1) = \dots = F(x_r)$ . By restricting to subfaces if necessary we can assume that  $x_i$  is in the relative interior of  $\sigma_i$ . Then all the  $x_i$  have the same distance to the  $(r-1)k$ -skeleton of  $\Delta_N$ .

Suppose all  $\sigma_i$  had dimension at least  $(r-1)k+1$ . Then these faces would involve at least  $r((r-1)k+2) = (r-1)(rk+2) + 2 > N+1$  vertices. Thus, one face  $\sigma_j$  has dimension at most  $(r-1)k$  and  $\text{dist}(x_j, \Delta_N^{((r-1)k)}) = 0$ . But then we have  $\text{dist}(x_i, \Delta_N^{((r-1)k)}) = 0$  for all  $i$ , so  $x_i \in \Delta_N^{((r-1)k)}$  and thus  $\sigma_i \subseteq \Delta_N^{((r-1)k)}$  for all  $i$ . This contradicts our assumption on  $f$ .  $\square$

If the topological Tverberg conjecture holds for  $r$  pairwise disjoint faces and dimension  $d+1$ , then it also holds for dimension  $d$  and the same number of faces; see Theorem 2.9. Thus, we are only interested in low-dimensional counterexamples. If  $r$  is not a prime power then the topological Tverberg conjecture fails for dimensions  $3r+1$  and above. Hence, the smallest counterexample this construction yields is a continuous map  $\Delta_{100} \rightarrow \mathbb{R}^{19}$  such that any six pairwise disjoint faces have images that do not intersect in a common point.

Define  $N_r(d)$  as the minimal integer  $N$  such that any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point. So far we know that for  $r$  a prime power we have  $N_r(d) = (r-1)(d+1)$  and for  $r$  not a prime power  $N_r(d) > (r-1)(d+1)$ . We will now establish some upper and lower bounds on the function  $N_r$ .

**Theorem 2.43.** *Let  $r \geq 2$  and  $d \geq 1$  be integers. Let  $q \geq r$  be a prime power and*

## 2.9 Counterexamples to the topological Tverberg conjecture

$N = (q-1)(d+1) - (q-r) = (q-1)d + r - 1$ . Then for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$  there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  with  $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$ .

*Proof.* Let  $M = (q-1)(d+1)$  and think of  $\Delta_N$  as a subcomplex of  $\Delta_M$ . Extend  $f$  to a continuous map  $F: \Delta_M \rightarrow \mathbb{R}^d$ . By the topological Tverberg theorem there are  $q$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_q$  of  $\Delta_M$  with  $F(\sigma_1) \cap \dots \cap F(\sigma_q) \neq \emptyset$ . Only  $q-r$  vertices of  $\Delta_M$  are not contained in  $\Delta_N$ , so at least  $r$  of the faces  $\sigma_1, \dots, \sigma_q$  are contained in  $\Delta_N$ .  $\square$

Let  $r \geq 6$  be an integer. By Bertrand's postulate there is a prime strictly between  $r-1$  and  $2r-4$ . Thus,  $N_r(d) \leq (2r-6)(d+1)$  for  $r \geq 6$ . We will now investigate the asymptotics of  $\frac{N_r(d)}{d}$  for  $d \rightarrow \infty$ .

**Lemma 2.44.** *Let  $d \geq 1$ ,  $r \geq 2$ ,  $k \geq 2$ , and  $N \geq (r-1)(d+1)$ . Suppose every continuous map  $F: \Delta_{k(N+1)-1} \rightarrow \mathbb{R}^{k(d+1)-1}$  has an  $r$ -fold Tverberg point. Then the same holds for any continuous map  $f: \Delta_N \rightarrow \mathbb{R}^d$ .*

*Proof.* The simplex  $\Delta_{k(N+1)-1}$  is isomorphic to the  $k$ -fold join  $\Delta_N^{*k}$ . Define  $F: \Delta_N^{*k} \rightarrow \mathbb{R}^{k(d+1)-1}$  by  $F(\lambda_1 x_1 + \dots + \lambda_k x_k) = (\lambda_1, \dots, \lambda_{k-1}, \lambda_1 f(x_1), \dots, \lambda_k f(x_k))$ . Then there are  $r$  pairwise disjoint faces  $\sigma_1, \dots, \sigma_r$  of  $\Delta_{k(N+1)-1}$  with  $F(\sigma_1) \cap \dots \cap F(\sigma_r) \neq \emptyset$ . Let  $\lambda_1^{(i)} x_1^{(i)} + \dots + \lambda_r^{(i)} x_r^{(i)} \in \sigma_i$  with  $F(\lambda_1^{(1)} x_1^{(1)} + \dots + \lambda_r^{(1)} x_r^{(1)}) = \dots = F(\lambda_1^{(r)} x_1^{(r)} + \dots + \lambda_r^{(r)} x_r^{(r)})$ . Thus,  $\lambda_j^{(1)} = \dots = \lambda_j^{(r)}$  for all  $j = 1, \dots, k$ , where we have equality for the  $\lambda_k^{(i)}$  since  $\lambda_k^{(i)} = 1 - \sum_{j=1}^{k-1} \lambda_j^{(i)}$ .

At least one  $\lambda_j^{(1)}$  is nonzero since  $\sum_j \lambda_j^{(1)} = 1$ . Then since  $\lambda_j^{(1)} = \dots = \lambda_j^{(r)}$  we have  $f(x_j^{(1)}) = \dots = f(x_j^{(r)})$  and these points come from pairwise disjoint faces in  $\Delta_N$ . Hence,  $f$  has an  $r$ -fold Tverberg point.  $\square$

Define  $\beta_r(d) = N_r(d) - (r-1)(d+1) \geq 0$ . The function  $\beta_r$  measures to which extent the topological Tverberg theorem fails in dimension  $d$  for a fixed  $r$ . We will now show that if the topological Tverberg theorem fails then it fails at least linearly in  $d$ .

**Corollary 2.45.** *For any  $r \geq 2$  and  $d, k \geq 1$  we have that  $N_r(k(d+1) - 1) \geq kN_r(d)$ . Moreover,  $\beta_r(k(d+1) - 1) \geq k\beta_r(d)$ . This can be written as  $\beta_r(d) \geq \frac{\beta_0}{d_0+1}(d+1)$  for  $d = k(d_0+1) - 1$  and  $\beta_0 = \beta_r(d_0)$  for some fixed  $d_0 \geq 1$ .*

*Proof.* Let  $N = N_r(d) - 1$  and  $f: \Delta_N \rightarrow \mathbb{R}^d$  a continuous map without Tverberg  $r$ -partition. Construct  $F: (\Delta_N)^{*k} \rightarrow \mathbb{R}^{k(d+1)-1}$  as above. Since  $F$  does not have a Tverberg  $r$ -partition we know that  $N_r(k(d+1) - 1)$  is at least one larger than the dimension of  $(\Delta_N)^{*k}$ , that is,  $N_r(k(d+1) - 1) \geq k(N+1) = kN_r(d)$ .

Moreover,  $N_r(k(d+1) - 1) \geq kN_r(d) = (r-1)(kd+k) + k\beta_r(d)$  and thus  $\beta_r(k(d+1) - 1) \geq k\beta_r(d)$ . Now,  $d = k(d_0+1) - 1$  implies  $k = \frac{d+1}{d_0+1}$ . Thus,  $\beta_r(d) = \beta_r(k(d_0+1) - 1) \geq k\beta_r(d_0) = \frac{d+1}{d_0+1} \cdot \beta_0$ .  $\square$

## 2 The constraint method and Tverberg-type results

**Lemma 2.46.** *Let  $r \geq 2$ . Suppose there is a  $d_0 \geq 1$  for which  $N_r(d_0) \geq \alpha_r \cdot (d_0 + 1) + c$  for some  $\alpha_r \geq r - 1$  and  $c \geq 0$ . Then  $N_r(d) \geq (\alpha_r + \frac{c}{d_0+1}) \cdot (d + 1)$  for  $d = k(d_0 + 1) - 1$  with  $k \in \{1, 2, \dots\}$ .*

*Proof.* We have that  $\beta_0 \geq (\alpha_r - (r - 1))(d_0 + 1) + c$ . Thus, by Corollary 2.45  $\beta_r(d) \geq (\alpha_r - (r - 1) + \frac{c}{d_0+1}) \cdot (d + 1)$  for  $d = k(d_0 + 1) - 1$ . This implies  $N_r(d) \geq (\alpha_r + \frac{c}{d_0+1}) \cdot (d + 1)$ .  $\square$

We can formulate this asymptotic behavior in the following way:

**Theorem 2.47.** *Let  $K_1, K_2, \dots$  be a sequence of simplicial complexes and denote the number of vertices of  $K_i$  by  $N_i$ . Let  $r$  be an integer that is not a power of a prime. If for every  $d \geq 1$  every continuous map  $f: K_d \rightarrow \mathbb{R}^d$  has an  $r$ -fold Tverberg point then for any  $\varepsilon > 0$  eventually  $N_d \geq (r - 1)d + \Omega(d^{1-\varepsilon})$ .*

## 2.10 Barycenters of polytope skeleta

The constraint method of forcing Tverberg partitions into a subcomplex is not tied to Tverberg-type results. Indeed, similar theorems can be proven for equivariant maps; see for instance [14, Section 3] for such results concerning mass partitions by hyperplanes. Here we will give another application of the constraint method of a slightly different flavor.

A problem originally motivated by mechanics is to determine whether each point in a polytope is the barycenter of points in an appropriate skeleton of the polytope [9]. Its resolution by Dobbins recently appeared in *Inventiones*.

**Theorem 2.48** (Dobbins [32]). *For any  $nk$ -polytope  $P$  and for any point  $p \in P$ , there are points  $p_1, \dots, p_n$  in the  $k$ -skeleton of  $P$  with barycenter  $p = \frac{1}{n}(p_1 + \dots + p_n)$ .*

Here we simplify the proof of this theorem by using the idea of equalizing distances of points to a certain unavoidable skeleton via equivariant maps to force them into the skeleton. Thus, we obtain the following slight generalization of Theorem 2.48.

**Theorem 2.49.** *Let  $P$  be a  $d$ -polytope,  $p \in P$ , and  $k$  and  $n$  positive integers with  $kn \geq d$ . Then there are points  $p_1, \dots, p_n$  in the  $k$ -skeleton  $P^{(k)}$  of  $P$  with barycenter  $p = \frac{1}{n}(p_1 + \dots + p_n)$ .*

More generally, one could ask for a characterization of all possibly inhomogeneous dimensions of skeleta and barycentric coordinates:

**Problem 2.50.** For given positive integers  $d$  and  $n$  characterize the dimensions  $d_1, \dots, d_n \geq 0$  and coefficients  $\lambda_1, \dots, \lambda_n \geq 0$  with  $\sum \lambda_i = 1$  such that for any  $d$ -polytope  $P$  there are  $n$  points  $p_1 \in P^{(d_1)}, \dots, p_n \in P^{(d_n)}$  with  $p = \lambda_1 p_1 + \dots + \lambda_n p_n$ .



Some remarks pertaining to this more general problem are contained in [32]. Still of greater generality is the problem of characterizing the subsets  $F \subseteq \mathbb{R}^d$  that contain a set of  $n$  not necessarily distinct vectors  $p_1, \dots, p_n \in F$  with  $p_1 + \dots + p_n = 0$ . In other words we are interested in those sets  $F$  which contain a not necessarily embedded  $(n-1)$ -simplex with barycenter at the origin. This is equivalent to  $F^n \subseteq \mathbb{R}^{d \times n}$  having non-empty intersection with  $W_n^{\oplus d}$ , where as before  $W_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0\}$ . If  $F$  is a set of dimension  $k$  with  $kn < d$ , then generically this intersection is empty for dimension reasons.

Let from now on the symmetric group  $\mathfrak{S}_n$  act on the space of matrices  $\mathbb{R}^{d \times n}$  by permuting columns.

**Theorem 2.51.** *Let  $n$  be prime,  $d \geq 1$  be an integer, and  $F \subseteq \mathbb{R}^d$  be closed. If there is an  $(n-2)$ -connected,  $\mathfrak{S}_n$ -invariant subset  $Q \subseteq W_n^{\oplus d} = \{(x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \sum x_i = 0\}$  such that for each  $(x_1, \dots, x_n) \in Q$  there is an  $i$  with  $x_i \in F$ , then there are  $p_1, \dots, p_n \in F$  with  $p_1 + \dots + p_n = 0$ .*

*Proof.* The map  $\Psi: Q \rightarrow \mathbb{R}^n, (x_1, \dots, x_n) \mapsto (\text{dist}(x_1, F), \dots, \text{dist}(x_n, F))$  is  $\mathfrak{S}_n$ -equivariant. Denote the diagonal by  $D = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_1 = \dots = y_n\}$ . The map  $\Psi$  induces by projection an  $\mathfrak{S}_n$ -equivariant map  $\Phi: Q \rightarrow D^\perp = W_n$ . The vector space  $W_n$  is  $(n-1)$ -dimensional and  $Q$  is  $(n-2)$ -connected, so  $\Phi$  has a zero by Lemma 2.3 applied to the subgroup  $\mathbb{Z}/n$  of  $\mathfrak{S}_n$ , which acts freely on  $W_n \setminus \{0\}$ . Let  $(p_1, \dots, p_n) \in Q$  with  $\Phi(p_1, \dots, p_n) = 0$ . This is equivalent to  $\text{dist}(p_1, F) = \dots = \text{dist}(p_n, F)$ . There is an index  $i$  such that  $p_i \in F$  and thus  $\text{dist}(p_i, F) = 0$ . Thus, all  $p_j$  satisfy  $\text{dist}(p_j, F) = 0$  and hence are in  $F$ , since  $F$  is closed. Since  $Q \subseteq W_n^{\oplus d}$  we have  $p_1 + \dots + p_n = 0$ .  $\square$

This theorem can be extended to the case that  $n$  is a prime power using a generalization of Dold's theorem to elementary Abelian groups; for a rather general version see [17].

For the proof of Theorem 2.49 we take for  $Q$  a skeleton of a certain polytope. Before proceeding we point out that the following naïve approach using a configuration space/test map scheme to prove Theorem 2.48 fails: the origin is the barycenter of  $n$  points in the  $d$ -skeleton of  $P$  if and only if the  $\mathfrak{S}_n$ -equivariant map  $(P^{(d)})^n \rightarrow \mathbb{R}^{dn}, (x_1, \dots, x_n) \mapsto \frac{1}{n}(x_1 + \dots + x_n)$  maps some point in  $(P^{(d)})^n$  to 0. However, for this map to be equivariant the  $\mathfrak{S}_n$ -action on  $\mathbb{R}^{dn}$  must be trivial, and thus  $\mathfrak{S}_n$ -equivariant maps  $(P^{(d)})^n \rightarrow \mathbb{R}^{dn} \setminus \{0\}$  exist. Dobbins's novel idea was to intersect with a test space in the domain to avoid this problem. We will employ that same idea. In contrast to Dobbins we need no requirement of general position. Thus, we also do not need any kind of approximation in the proof.

*Proof of Theorem 2.49.* We can assume that  $p = 0$  is in the interior of  $P$ , otherwise we could restrict to a proper face of  $P$  with the origin in its relative interior. Let first  $n$  be prime. Consider again the linear space  $W_n^{\oplus d} = \{(x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \sum x_i = 0\}$  of codimension  $d$ .

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Then  $C = P^n \cap W_n^{\oplus d}$  is a polytope of dimension  $(n-1)d$ . The  $(n-1)$ -skeleton  $C^{(n-1)}$  of  $C$  is homotopy equivalent to a wedge of  $(n-1)$ -spheres and thus is  $(n-2)$ -connected.

Let  $(x_1, \dots, x_n) \in C^{(n-1)}$ . By Theorem 2.51 we need to show that one  $x_i$  lies in  $P^{(k)}$ . Suppose for contradiction that  $x_i \notin P^{(k)}$  for all  $i = 1, \dots, n$ . For each  $x_i$  let  $\sigma_i$  be the inclusion-minimal face of  $P$  with  $x_i \in \sigma_i$ . We have that  $\dim \sigma_i \geq k+1$ . Each face of  $C$  is of the form  $(\tau_1 \times \dots \times \tau_n) \cap W_n^{\oplus d}$  with the  $\tau_i$  faces of  $P$ . The point  $(x_1, \dots, x_n)$  lies in the face  $(\sigma_1 \times \dots \times \sigma_n) \cap W_n^{\oplus d}$  but in no proper subface. The dimension of this face is  $\dim((\sigma_1 \times \dots \times \sigma_n) \cap W_n^{\oplus d}) \geq n(k+1) - d \geq n$ . Thus,  $(\sigma_1 \times \dots \times \sigma_n) \cap W_n^{\oplus d} \notin C^{(n-1)}$ , which is a contradiction.

The case for general  $n$  follows by a simple induction with respect to the number of prime divisors, as in [32]: Suppose  $n = q_1 \cdots q_t$  with  $q_i$  prime and the theorem holds for any number  $n$  that is a product of at most  $t-1$  primes. Let  $m = q_2 \cdots q_t$ . Since  $m \cdot q_1 k = nk \geq d$ , there are  $m$  points  $x_1, \dots, x_m$  in  $P^{(q_1 k)}$  with  $p = \frac{1}{m}(x_1 + \dots + x_m)$ . Each  $x_i$  is contained in a  $(q_1 k)$ -face  $\sigma_i$  of  $P$ . Thus, there are  $q_1$  points  $y_i^{(1)}, \dots, y_i^{(q_1)}$  in the  $k$ -skeleton  $\sigma_i^{(k)}$  of  $\sigma_i$  with  $x_i = \frac{1}{q_1}(y_i^{(1)} + \dots + y_i^{(q_1)})$ . In particular, the  $y_i^{(j)}$  are also contained in  $P^{(k)}$  and  $p = \frac{1}{m} \sum_{i=1}^m \frac{1}{q_1} \sum_{j=1}^{q_1} y_i^{(j)} = \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^{q_1} y_i^{(j)}$ .  $\square$

# 3 Combinatorial restrictions on manifold triangulations

## Abstract

We give a complete combinatorial and topological classification of pseudo-manifolds that are positively curved in a combinatorial sense. We simplify the proof of a result of Brady, McCammond, and Meier [21] that any closed and orientable 3-manifold admits a triangulation with edge-degrees four, five, and six. We investigate relations to geometric structures in dimension three and prove higher-dimensional versions.

**Publication remark.** This chapter is based on joint work with Frank Lutz and John M. Sullivan [43]. The non-polytopal triangulation in Section 3.4 was presented in [41].

A *triangulation* of a topological space  $X$  is a simplicial complex whose geometric realization is homeomorphic to  $X$ . Thus, a manifold triangulation is a simplicial complex with geometric realization a manifold. A manifold  $M$  is said to have *piecewise linear structure* (or just  $M$  is PL for short) if  $M$  has an atlas of charts, where the transition functions are piecewise affine as maps of  $\mathbb{R}^d$ . Any PL manifold structure induces a triangulation, but not every triangulation of a manifold is PL, that is, not every triangulation comes from a PL structure; see Edwards [35]. A triangulation of a  $d$ -sphere is PL if and only if it has a common subdivision with the boundary of the standard simplex; see Pachner [76]. A triangulation of a  $d$ -dimensional manifold is PL if and only if all vertex links are PL triangulations of the  $(d-1)$ -sphere. For  $d = 4$  it is an open problem whether there are triangulated 4-spheres that are 4-dimensional PL manifold triangulations but that do not have a common subdivision with the boundary of the 5-simplex.

Every smooth manifold can be equipped with a PL-structure, but not every topological manifold is PL; see Kirby and Siebenmann [55]. A first example of a non-triangulable manifold can be obtained by combining work of Freedman [39] and Casson; see Akbulut and McCarthy [1]. Due to Perelman's proof of the Poincaré conjecture it in particular follows that all non-smoothable 4-manifolds do not admit a triangulation. Recently, Manolescu [68] showed that there also exists a  $d$ -manifold that does not admit a triangulation for every  $d \geq 5$ .

### 3 Combinatorial restrictions on manifold triangulations

In the following we will be interested in investigating one of the simplest combinatorial analogues of curvature imaginable: the number of facets a face of codimension two is contained in. We will refer to faces of dimension  $d - 2$  in a  $d$ -complex as *subridges*, and the number of facets it is contained in is its *valence*. Which bounds on valence correspond to positive curvature depends on the dimension. Given a regular  $d$ -simplex the dihedral angle between any two of its  $(d - 1)$ -faces is  $\arccos(\frac{1}{d})$ ; see Parks and Wills [77]. Thus, since  $\arccos(\frac{1}{3}) < \frac{2\pi}{5}$  in dimensions two and three, valence five leads to an angle defect around the subridge, while in higher dimensions the valence of a subridge needs to be at most four to correspond to an angle defect.

A  $d$ -dimensional manifold triangulation with valences bounded from above by five for  $d \leq 3$  or valences bounded from above by four for  $d \geq 4$  will be called *combinatorially positively curved*. For the case  $d = 3$  manifold triangulations with positive combinatorial curvature were studied by Trout [95] by combinatorial means. Trout shows that this notion of curvature leads to combinatorial versions of the Bonnet–Myers theorem, that is, a diameter bound, and classifies the complexes with maximal combinatorial diameter. We give a simple proof of the fact that combinatorially positively curved triangulations have bounded volume via inducing a metric on every facet of the triangulation and then using results from metric geometry.

We show that any PL-manifold admits a triangulation with valences bounded by nine. Moreover, we investigate relations between valences and geometric structures on the underlying manifold in dimension  $d = 3$ . We focus on the case  $d = 3$  throughout and explain the construction of some examples of particularly round triangulations of  $S^3$ .

The number of  $i$ -faces of a simplicial complex will be denoted by  $f_i$ . We collect these numbers in the  $f$ -vector  $(f_0, \dots, f_d)$ . For a given number  $f_0$  of vertices, triangulations of  $S^3$  with the least number  $f_3$  of facets are the stacked instances. To *stack* a facet means to remove it from the triangulation and cone the resulting boundary with a new vertex. A triangulation is *stacked* if it can be obtained from the boundary of the simplex by repeated stackings. These triangulations minimize the  $f$ -vector  $(f_0, \dots, f_d)$  among all pseudomanifolds with the same number of vertices by the lower bound theorem [93]. On the other end of this spectrum are the neighborly triangulations of the 3-sphere: they maximize the  $f$ -vector coordinatewise among all triangulations of  $S^3$  by the upper bound theorem; see McMullen [71] and Stanley [88]. A triangulation of a  $d$ -manifold is called *neighborly* if it has a complete  $\lfloor \frac{d-1}{2} \rfloor$ -skeleton. Boundary complexes of cyclic polytopes are instances of neighborly triangulations.

Stacked and neighborly 4-polytopes are far from round in the sense that they have edges of extremal valence. Stacked 3-dimensional triangulations always have an edge of valence three, and every stacked triangulation on more than seven vertices has an edge of valence

### 3.1 From valence bounds to volume bounds

greater than five. Vertex links in neighborly 4-polytopes are stacked 2-spheres: the convex hull of all but one vertex  $v$  of a neighborly 4-polytope (with vertices in general position) induces a triangulation of  $\text{lk}(v)$  without interior edges. This defines a stacking order for  $\text{lk}(v)$ . This argument is due to Perles and can be found in Altshuler and Steinberg [2]. Thus, neighborly 4-polytopes always have an edge of valence three. On the other hand they also have edges of large valence.

## 3.1 From valence bounds to volume bounds

A  $d$ -pseudomanifold is a pure  $d$ -dimensional simplicial complex, where vertex links are connected for  $d > 1$ , every  $(d-1)$ -face (*ridge*) is contained in precisely two facets, and such that any two facets can be connected by a facet-ridge path, that is a path in the 1-skeleton of the dual complex. This last property is called *strong connectivity*. If instead of every ridge being contained in precisely two facets, every ridge is contained in one or two facets, we call the complex a *pseudomanifold with boundary*. The *boundary* is the subcomplex induced by ridges contained in exactly one facet.

All 0-pseudomanifolds are  $S^0$ , 1-pseudomanifolds are triangulated circles, and thus 2-pseudomanifolds are triangulated surfaces. We give a complete combinatorial (and hence topological) classification of all pseudomanifolds of positive combinatorial curvature. The special cases of low dimensions  $d = 2, 3$  will be postponed to Section 3.3. The enumeration of 3-pseudomanifolds with positive combinatorial curvature is achieved with a computer. First we will show that positive combinatorial curvature leads to a volume bound in terms of the number of facets.

Constant valence triangulations of spheres are always realizable as boundary complex of a regular polytope. The  $d$ -simplex and the  $d$ -crosspolytope are the only two regular simplicial polytopes that exist in dimensions  $d \geq 5$ . (Their boundary spheres have dimension  $d - 1$ .) The simplex has constant valence three, and the crosspolytope has constant valence four. In low dimensions  $d = 3, 4$  there is one more regular simplicial polytope, respectively, corresponding to constant valence five: the icosahedron and the 600-cell. Such examples only exist in low dimensions since here the dihedral angles of the regular facets are less than  $\frac{2\pi}{5}$  and, thus, five facets around a common subridge lead to an angle defect.

**Lemma 3.1.** *Let  $d \geq 2$ . There is a spherical  $d$ -simplex with all dihedral angles equal to  $\frac{2\pi}{3}$ . Also, there is a spherical  $d$ -simplex with all dihedral angles equal to  $\frac{2\pi}{4}$ . For  $d = 2$  and  $d = 3$  there is a spherical  $d$ -simplex with dihedral angles  $\frac{2\pi}{5}$ . Moreover, all these simplices are realized with their full group of combinatorial symmetries.*

*Proof.* The regular simplicial polytopes can be realized with their full group of symmetries

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and with vertices on the unit sphere. Radially projecting the boundary of the polytope onto the unit sphere gives the desired spherical simplices.  $\square$

To transfer from combinatorics to geometry we will induce a metric on every facet of a triangulation and then use the toolbox of metric comparison geometry; see Burago, Burago, Ivanov [24] for an introduction. In particular, we will use the following lemma:

**Lemma 3.2** (Burago, Gromov, and Perelman [25, 2.9(6)]). *Let  $K$  be a  $d$ -pseudomanifold. Define on every facet of  $K$  the metric of a simplex from a space of constant curvature  $k$  such that the angle around every subridge is at most  $2\pi$ . Then  $K$  is an Alexandrov space of curvature  $\geq k$ .*

Notice that while in [25] this lemma is stated for any simplicial complex, where every ridge is contained in two facets and facets come from the space of constant curvature  $k$ , Burago, Gromov, and Perelman actually use that vertex links are connected for  $k > 0$ .

That valence bounds lead to volume bounds is a purely combinatorial observation for 2-dimensional triangulations: here combinatorics and geometry are tightly related via the Euler characteristic. In the trivial 1-dimensional case valence bounds the number of facets (edges) that the empty set is contained in. Hence, it is a global bound on the number of edges.

Let  $f = (f_0, f_1, f_2)$  be the  $f$ -vector of a triangulated surface  $M$  of Euler characteristic  $\chi(M)$ . By Euler's equation,  $f_0 - f_1 + f_2 = \chi(M)$ , and by double counting of the edge-triangle incidences,  $2f_1 = 3f_2$ , the  $f$ -vector of  $M$  can be written as  $f = (f_0, 3f_0 - 3\chi(M), 2f_0 - 2\chi(M))$ .

A valence bound in dimension two bounds the *degree*  $\deg(v)$  of a vertex  $v$ , that is, the number of incident edges or, equivalently, the number of incident triangles. If  $\deg(v) \leq t$  for all vertices  $v$ , then by double counting  $2f_1 = \sum \deg(v) \leq f_0 t$ . Thus,  $(6 - t)f_0 \leq 6\chi(M)$ . In particular, if  $t = 5$  then  $f_0 \leq 6\chi(M) \leq 12$  and  $\chi(M)$  is positive, that is,  $M$  is one of the two spherical surfaces  $S^2$  or  $\mathbb{R}\mathbf{P}^2$ . The largest such triangulation is the boundary of the icosahedron, and the only triangulation with valence bounded by five of  $\mathbb{R}\mathbf{P}^2$  is the antipodal quotient of the icosahedral triangulation. See Section 3.3 for a complete list of surface triangulations with valence bounded by five.

As mentioned in the introduction this counting argument above works because of the lower bound  $f_1 \geq 3f_0 - 6$  in dimension two. In higher dimensions the lower bound  $f_1 \geq (d+1)f_0 - \binom{d+2}{2}$  for manifold triangulations is due to Walkup [103] for dimensions  $d = 3$  and 4 and due to Kalai [53] for general  $d$ . This was generalized to pseudomanifolds by Tay [93]. Now if the average vertex degree is bounded from above by  $2d+2-\varepsilon$  for some  $\varepsilon > 0$ , we obtain  $2d+2-\varepsilon \geq \frac{2f_1}{f_0} = 2d+2 - \frac{1}{f_0} \binom{d+2}{2}$ , and thus  $f_0 \leq \frac{1}{\varepsilon} \binom{d+2}{2}$ . Specializing to  $d = 3$  Walkup showed that if  $K$  is a 3-manifold triangulation with  $f_1 < 4f_0$  then  $K$  is homeomorphic

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to  $S^3$ . (For  $f_1 \leq 4f_0$  additionally  $S^1 \times S^2$  and  $S^2 \times S^1$  are possible.) This was reinterpreted in terms of valence bounds by Luo and Stong [64]: for a 3-manifold triangulation  $M$  with  $f$ -vector  $(f_0, f_1, f_2, f_3)$  define the *average valence* of edges as  $\bar{v} = \frac{6f_3}{f_1}$  and the *average vertex degree*  $\bar{z} = \frac{2f_1}{f_0}$ . By double counting we obtain  $f_3 = 2f_2$  and  $f_3 = f_1 - f_0$ , since  $0 = \chi(M) = f_0 - f_1 + f_2 - f_3$ ; hence  $f_1 < 4f_0$  is equivalent to  $\bar{v}$  being bounded above by 4.5. Since  $\bar{v} = \frac{f_3}{f_1} = \frac{6f_1 - 6f_0}{f_1} = 6 - \frac{12}{\bar{z}}$  it is also equivalent to  $\bar{z} < 8$ .

A strict average valence bound of 4.5 leads to a volume bound and, moreover, the triangulation is homeomorphic to  $S^3$ , while there are arbitrarily large triangulations with average valence equal to 4.5 (arbitrarily large coverings of the triangulations of  $S^2 \times S^1$ ). Here we obtain a volume bound if the valence of every edge is at most five. That this statement is not true for the average valence shows that a counting argument like for dimension two will not work for dimension three.

**Theorem 3.3.** *Let  $d \geq 2$ . Let  $K$  be a  $d$ -pseudomanifold with valences bounded by four. Then  $K$  has at most  $2^{d+1}$  facets. If  $d = 3$  and  $K$  has valences bounded by five, then  $K$  has at most 600 facets. If  $d = 2$  and  $K$  has valences bounded by five, then  $K$  has at most 20 triangles.*

That a triangulation of a 3-manifold with edge valences bounded by five is always finite was already noted by Stone [89]. For the proof we will need the following lemma from metric geometry. Here  $\mu_n$  denotes the  $n$ -dimensional Hausdorff measure.

**Lemma 3.4** (Burago, Burago, and Ivanov [24, Corollary 10.6.9]). *Let  $X$  be an  $n$ -dimensional Alexandrov space of curvature bounded below by one. Then  $\mu_n(X) \leq \mu_n(S^n)$ .*

*Proof of Theorem 3.3.* Induce the correct metric from Lemma 3.1 on every facet of the triangulation, that is, the metric of the regular spherical  $d$ -simplex with dihedral angles  $\frac{2\pi}{k}$  if the valences in  $K$  are bounded by  $k$ . This metric agrees on lower-dimensional faces since the simplices are realized with their full group of (transitive) symmetries. This turns the triangulation into an Alexandrov space with curvature bounded below by one by Lemma 3.2. Now by Lemma 3.4 the complex  $K$  has as most as many facets as the corresponding regular simplicial polytope.  $\square$

## 3.2 A combinatorial classification

We give a complete combinatorial classification of pseudomanifolds with valences bounded by four. The case of valences bounded by three is essentially trivial.

**Lemma 3.5.** *For  $d \geq 1$ , the only  $d$ -pseudomanifold  $K$  with the property that every subridge has valence at most three is the boundary  $\partial\Delta_{d+1}$  of the  $(d+1)$ -simplex  $\Delta_{d+1}$ .*

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This is simple to see combinatorially and a rephrasing of the fact that a simple pseudomanifold is the boundary of the simplex: by induction over  $d$  every vertex link is the boundary of a simplex. Deleting any vertex and all its incident faces leaves a pseudomanifold with boundary, where every boundary subridge has valence one. Hence, this deletion is a simplex. Thus,  $K$  is the boundary of a simplex. We will, however, now present a geometric proof to highlight the geometric nature of valence bounds in a simple case.

*Proof.* Let  $K$  be a  $d$ -pseudomanifold with valences equal to three. Give every facet the spherical metric with dihedral angles  $\frac{2\pi}{3}$  from Lemma 3.1. Since valences are equal to three, this defines a spherical metric on  $K$ . In particular,  $K$  is an Alexandrov space with curvature bounded below by one. By Lemma 3.4  $K$  has at most  $d+1$  facets and hence is the boundary of a simplex.  $\square$

**Theorem 3.6.** *Let  $K$  be a  $d$ -pseudomanifold,  $d \geq 1$ , with the property that every subridge has valence at most four. Then  $K$  is the join product of boundaries of simplices, namely exactly of the boundaries of empty faces in  $K$ . In particular,  $K$  is the boundary of a  $(d+1)$ -polytope.*

This theorem (with the prerequisite that  $K$  be a simplicial polytope) is due to Deza, Dutour, and Shtogrin [31, Theorem 2.5]. Here we present a different and self-contained proof.

*Proof.* A 1-pseudomanifold is a triangulated circle. Therefore the only examples meeting the valence condition are the boundary of a triangle and the boundary of a square, which is the join product  $S^0 * S^0$ . So let us assume that  $d \geq 2$  and that the statement holds for all  $m$ -dimensional pseudomanifolds  $K$  with  $m < d$ .

First, we claim that if  $K$  contains a missing edge, that is, two vertices  $v$  and  $w$  that are not connected by an edge, then  $K$  is the suspension  $\{v, w\} * L$ , where  $L = \text{lk}(v)$ . Thus, by induction  $K$  is a join product of boundaries of simplices. This claim follows from the fact that deleting  $v$  yields a pseudomanifold  $K'$  with boundary and at least one interior vertex  $w$  such that the subridges in the boundary have valence at most two. Such a simplicial complex necessarily is  $w * L$ , where  $L$  is the boundary: start with any  $(d-1)$ -face  $\sigma$  in  $L$  with  $u * \sigma \in K'$  for some interior vertex  $u \in K'$ . Any  $(d-1)$ -face  $\tau$  in  $L$  sharing a common  $(d-2)$ -face with  $\sigma$  is contained in at most two facets of  $K'$ . Thus,  $u * \tau$  must be this facet. By induction using the strong connectivity of  $L$  we get  $K' = u * L$  and in particular  $u = w$ .

If the  $d$ -complex  $K$  has a complete  $(d-1)$ -skeleton, then  $K$  is the boundary of the  $(d+1)$ -simplex  $\Delta_{d+1}$ . So let us assume that  $K$  has a complete  $k$ -skeleton, but not a complete  $(k+1)$ -skeleton, with  $1 \leq k < d-1$ . It follows that  $K$  has at least one *empty*  $(k+1)$ -simplex  $\sigma \subseteq \{1, \dots, n\}$ , that is,  $\sigma$  is not a face of  $K$ , but all the facets of  $\sigma$  are faces



### 3.3 Valence bounds in low dimensions

of  $K$ . We will show that every facet of  $\sigma$  has the same link  $L$  in  $K$ , from which it follows that  $K = \partial\sigma * L$ .

Let  $\tau$  be a ridge of the boundary  $\partial\sigma$  of  $\sigma$  such that  $F = v * \tau$  and  $G = w * \tau$  are the two maximal faces of  $\partial\sigma$  containing  $\tau$ . Now,  $v$  and  $w$  are vertices of  $\text{lk}(\tau)$  that are not connected (if they were, then  $\sigma$  would be a face of  $K$ ). Thus, as in the above case of a missing edge,

$$L := \text{lk}_K(F) = \text{lk}_{\text{lk}_K(\tau)}(v) \stackrel{!}{=} \text{lk}_{\text{lk}_K(\tau)}(w) = \text{lk}_K(G)$$

and

$$\text{lk}_K(\tau) = \{v, w\} * L.$$

Since we can reach every maximal face of  $\partial\sigma$  from  $F$  along ridges, the maximal faces of  $\partial\sigma$  all have  $L$  as their link; thus  $K = \partial\sigma * L$ .

Lastly, notice that a join of boundaries of simplices is the boundary of a simplicial polytope.  $\square$

Boundaries of simplicial polytopes are PL triangulations of spheres. All triangulations of closed 3-manifolds are PL; see Moise [74]. Thus, we obtain the following:

**Corollary 3.7.** *Every combinatorially positively curved manifold triangulation is PL.*

While it is reasonable to expect that all combinatorially positively curved manifold triangulations are quotients of boundaries of polytopes (and this is true in all dimensions but  $d = 3$ ), this is in fact wrong, see Theorem 3.19.

### 3.3 Valence bounds in low dimensions

So far we have given a complete combinatorial classification of pseudomanifolds with valences bounded by four. In dimensions two and three there are finitely many manifold triangulations with valences bounded by five. A combinatorial classification in dimension two is quite simple (as there are only twelve examples with at most twelve vertices) but such a classification has not been done explicitly. However, much more general classes of surface triangulations have been enumerated. For example, Brinkmann and McKay [23] enumerated all triangulations of  $S^2$  with at most 23 vertices.

**Theorem 3.8** (Brinkmann and McKay [23]). *There are precisely twelve triangulated surfaces with vertex-degree at most five, eleven spheres and the vertex-minimal triangulation  $\mathbb{RP}_6^2$  of the projective plane.*

These triangulations are the tetrahedron, triangular bipyramid, a tetrahedron where two triangles have been stacked, octahedron, an octahedron where one face has been stacked, an

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octahedron where two opposite faces have been stacked (this can be thought of as  $\frac{3}{5}$ th of the icosahedron and we will make this notion precise below), suspension of a pentagon, a triangulated cube, the dual of the associahedron,  $\frac{4}{5}$ th of the icosahedron, the icosahedron itself, and the antipodal quotient of the icosahedron. Since 2-pseudomanifolds are triangulated surfaces, this immediately implies:

**Corollary 3.9.** *There are precisely twelve 2-pseudomanifolds with vertex-degree at most five, eleven spheres and the vertex-minimal triangulation  $\mathbb{RP}_6^2$  of the projective plane.*

This leaves the task of classifying the combinatorial types of 3-pseudomanifolds with edge valences bounded by five. By Theorem 3.3 they have at most 600 facets. Since this is quite a large number of facets the enumeration of all instances will necessarily require the help of a computer. This was achieved by Frank Lutz with a GAP implementation. See [43] for details of the algorithm.

**Theorem 3.10** ([43]). *There are exactly 4787 combinatorially distinct triangulated 3-manifolds with edge valence at most five: 4761 triangulations of  $S^3$ , 22 of  $\mathbb{RP}^3$ , two of  $L(3, 1)$ , one of  $L(4, 1)$ , and one of the cube space  $S^3/Q$ . In particular, all examples are spherical.*

The topological types of the 4787 distinct combinatorial types were recognized with the bistellar flip program BISTELLAR [66] (see Björner and Lutz [12] for a description). The largest of the examples is the 600-cell with 120 vertices. See Section 3.4 for some interesting examples and general constructions.

We saw that every 2-pseudomanifold with vertex-degree at most five is either one of eleven 2-spheres or the vertex-minimal triangulation  $\mathbb{RP}_6^2$  of the projective plane. It follows, that in every 3-pseudomanifold  $K$  has the eleven 2-spheres and  $\mathbb{RP}_6^2$  as possible vertex links. Here we use that vertex links are connected. If  $\mathbb{RP}_6^2$  does not appear as a vertex link, then  $K$  is a triangulated 3-manifold and therefore one of the 4787 examples above.

**Theorem 3.11.** *There are exactly 41 distinct 3-pseudomanifolds with edge valence at most five that have  $\mathbb{RP}_6^2$  as one of their vertex links. All 41 examples are homeomorphic to the suspension  $S^0 * \mathbb{RP}^2$ .*

This is proved like Theorem 3.10 with the help of a computer.

**Corollary 3.12.** *There are exactly 4828 distinct 3-pseudomanifolds with edge valence at most five, 4787 manifolds and 41 proper pseudomanifolds.*

## 3.4 Three-dimensional examples

In this section we describe some interesting examples of 3-dimensional triangulations with valence bounded by five. We will give some general construction principles for 3-dimensional

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triangulations: extended bipyramids and Hopf lifts.

A metric  $d$  on a manifold  $M$  obtained from gluing  $M$  from spherical simplices by isometries of their faces is called a *spherical cone-manifold structure* on  $M$ . We call  $(M, d)$  a *spherical cone-manifold* and refer to  $d$  as a (*spherical*) *cone metric*. See Cooper, Hodgson, and Kerckhoff [27] for an introduction. Such a metric on  $M$  is a spherical metric away from the subridges with an angle  $\neq 2\pi$  around them. The collection of these subridges is called the *singular locus*. The angle around such a singular subridge is the *cone angle*.

We will always assume that a cone metric comes from a triangulation (in the sense of a simplicial complex) and a spherical metric on every facet. This is without loss of generality. We will refer to a triangulation with a spherical metric on every facet as a *metric triangulation* of the induced cone metric. Given a cone metric one can ask for particularly symmetric metric triangulations. A triangulation  $K$  of a spherical cone-manifold  $M$  is called *regular metric triangulation* if there exists a regular spherical simplex such that inducing its metric on every facet of  $K$  gives the cone metric on  $M$ .

Given a cone metric on  $S^{d-1}$  we can define a natural suspension cone metric on  $S^d$ . Glue the cone-manifold structure on  $S^{d-1}$  from spherical  $(d-1)$ -simplices. For every spherical  $(d-1)$ -simplex  $\sigma$  the cone over  $\sigma$  is naturally a spherical  $d$ -simplex; see Burago, Burago, and Ivanov [24, Sec. 3.6]. Realize  $\sigma$  on the equator of the round  $d$ -sphere with metric diameter  $\pi$ . The cone over  $\sigma$  is the spherical  $d$ -simplex that consists of all shortest paths connecting  $\sigma$  to the north pole. Use this metric on the facets of the suspension to obtain a cone metric on  $S^d$ . This metric is called *spherical suspension metric*. It is a cone metric by construction, since it comes with a triangulation that simply is the suspension of the triangulation of  $S^{d-1}$  inducing the cone metric.

**Theorem 3.13.** *Let  $d$  be a spherical cone metric on  $S^2$  that comes from inducing the metric of the spherical triangle with dihedral angles  $\frac{\pi}{2}$  (respectively  $\frac{2\pi}{5}$ ) on every facet of a triangulation  $K'$  of  $S^2$ . Then the spherical suspension metric on  $S^3$  admits a regular metric triangulation.*

If edge lengths (and thus dihedral angles) are  $\frac{\pi}{2}$  in  $K'$ , then this follows trivially from the definition of spherical suspension metric: the regular metric triangulation is the (combinatorial) suspension (bipyramid)  $K = \Sigma K'$ . Inducing the metric with all edge lengths equal to  $\frac{\pi}{2}$  on  $K$  gives precisely the spherical suspension metric. For dihedral angles of  $\frac{2\pi}{5}$  in  $K'$  this construction fails, since then the edge lengths in  $K$  are not all equal. The correct metric on each facet is the metric from the suspension of the icosahedron, which does not consist of regular simplices. Thus, in this case we have to define an extended analogue of the suspension.

Before we prove Theorem 3.13 let us consider the example where  $K'$  is the boundary of the

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icosahedron and hence the induced metric on  $S^2$  is the round metric. The spherical metric suspension is the round metric on  $S^3$ . This metric has three regular metric triangulations: the boundary of the simplex, crosspolytope, and 600-cell.

*Proof of Theorem 3.13.* We remarked that the case of dihedral angles  $\frac{\pi}{2}$  is obvious by construction. Now let every dihedral angle in  $K'$  be  $\frac{2\pi}{5}$ . Project the 600-cell onto the unit sphere to obtain a regular metric triangulation  $T$  of the round metric on  $S^3$ . Let  $v$  be a vertex in  $T$  and  $\sigma$  a triangle in  $\text{lk}(v)$ . The collection of great semi-circles that connect  $v$  to  $-v$  and pass through a point in  $\sigma$  is a subspace (but not a subcomplex) of  $T$  called a *slice of the 600-cell*. Such a slice consists of facets and half-facets such that mirroring half-facets gives a proper tetrahedron. The boundary of the slice is cut into three parts by geodesics of edges connecting  $v$  to  $-v$ . Each part of the boundary corresponds to an edge of  $\sigma$ .

Define a metric triangulation  $K$  in the following way: for every triangle of  $K'$  pick a copy of a slice of the 600-cell. For any edge in  $K'$  glue the slices together along the corresponding part of their boundary. This identifies half-facets to full tetrahedra and  $K$  is a triangulation. Together with the metric information from  $T$  it is a regular metric triangulation of the spherical suspension metric.  $\square$

In the proof above we constructed a triangulation  $K$  of  $S^3$  for every triangulation  $K'$  of  $S^2$ . The triangulation  $K$  is called the *extended bipyramid* over  $K'$  and denoted  $\Sigma_{\text{ext}}K'$ . If  $K'$  has  $f$ -vector  $(k + 2, 2k, 3k)$  then  $\Sigma_{\text{ext}}K'$  has  $f$ -vector  $(11k + 10, 71k + 10, 120k, 60k)$ . In particular, an extended bipyramid on  $f_0$  vertices exists if and only if  $f_0 - 10$  is at least 22 and divisible by eleven.

A purely combinatorial way to construct the extended bipyramid of  $K'$  is to start with a vertex  $v$  with vertex link isomorphic to  $K'$ . Now extend this triangulation by the rule that every edge in the link of  $v$  have valence five. Then at combinatorial distance two from  $v$  we see a triangulated sphere that combinatorially is the dual of  $K'$ , where every 2-face has been stacked once. One can continue to build  $\Sigma_{\text{ext}}K'$  via the rule that every edge is in as many facets as possible but at most five; see Trout [95, Sec. 5].

The metric suspension of any Alexandrov metric on  $S^2$  of curvature bounded below by one has diameter  $\pi$ . The two suspension vertices in an extended bipyramid are always connected by a geodesic that consists of edges. Thus, the combinatorial diameter of an extended bipyramid is five, since the edge length of the spherical simplex with dihedral angles  $\frac{2\pi}{5}$  is  $\frac{\pi}{5}$ . This characterizes extended bipyramids among all positively curved 3-manifold triangulations.

**Theorem 3.14** (Trout [95]). *A combinatorially positively curved 3-dimensional triangulation  $K$  has combinatorial diameter at most five and is an extended bipyramid if and only if it has combinatorial diameter five.*

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This is the combinatorial analogue of the fact that an Alexandrov metric on  $S^3$  of curvature bounded below by one has diameter  $\pi$  if and only if it is the metric suspension of such an Alexandrov metric on  $S^2$ .

For every cone metric on  $S^2$  one can obtain a cone metric on  $S^3$  by suspending. A second way to construct a cone metric on  $S^2$  from a given metric on  $S^2$  is to lift through the Hopf fibration. Think of  $S^3$  as the unit sphere in  $\mathbb{C}^2$  and think of  $S^2$  as the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ . The map  $h: S^3 \rightarrow S^2, (w, z) \mapsto \frac{w}{z}$  with  $\frac{w}{0} := \infty$  is called the *Hopf fibration*. It is a fiber bundle with base  $S^2$  and fiber  $S^1$ . The rotations  $R_\varphi: S^3 \rightarrow S^3, (w, z) \mapsto (e^{i\varphi}w, e^{i\varphi}z)$  leave the fibers invariant.

The link type of the singular locus and the cone angles alone do not uniquely determine the cone metric. If a cone metric is uniquely determined by this data it is called *rigid*. In several papers the volume and rigidity of cone metrics have been studied; see for instance Kolpakov and Mednykh [59], Kolpakov [57, 58], Mednykh [72], and Weiss [104]. A particular simple way to determine the volume of a cone-manifold is to find a regular metric triangulation of it and count the number of facets as well as determine the volume of a regular spherical simplex.

A Hopf linked singular locus can arise by lifting a cone-metric on  $S^2$  via the Hopf fibration  $h$ . Then the fibers over the cone points on  $S^2$  are the singular locus with the same cone angle in  $S^3$  as the projected point in  $S^2$ . To lift a cone metric through the Hopf fibration we are going to define the metric on  $S^3$  by first partially lifting triangles and then gluing the metric together. Given a spherical triangle  $\sigma \subseteq S^2$ , the solid torus  $h^{-1}(\sigma) \subseteq S^3$  inherits a metric from the round 3-sphere. It will in general not be possible to glue the solid tori  $h^{-1}(\sigma) - \sigma$  ranging over the triangles of a metric triangulation  $K'$  of a cone metric on  $S^2$  – according to the edges of  $K'$ . Rather there is a closing condition coming from the twist of the bundle and to satisfy it we need to shorten the fibers in  $h^{-1}(\sigma)$ . We want that the length of fibers is half the area of  $K'$ ; see Sullivan [90] for details. The resulting cone metric on  $S^3$  is called the *Hopf lift* of the cone metric on  $S^2$ . Again, we want to find a good combinatorial model for this metric.

**Theorem 3.15.** *Let  $K'$  be a triangulation of  $S^2$  and induce the metric of the spherical triangle with all dihedral angles equal to  $\frac{2\pi}{5}$  on each facet of  $K'$ . The Hopf lift of the resulting cone metric admits a regular metric triangulation. (This triangulation is not a simplicial complex if  $K'$  is the boundary of the tetrahedron.)*

To prove this theorem we first need to find a suitable triangulation of a solid torus and then glue these solid tori together to obtain a triangulation of  $S^3$ . The triangulation  $B(n) = \{[k, k+1, k+2, k+3] \subseteq \mathbb{Z}/n \mid k \in \mathbb{Z}/n\}$ ,  $n \geq 7$ , is called the *Boerdijk–Coxeter helix* [29]. We will restrict to the case where  $n$  is divisible by three. Then there are three

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closed curves of valence one determined by the vertices  $k, k + 3, k + 6, \dots$  for  $k = 0, 1, 2$ , respectively. These cut  $\partial B(n)$  into three annuli bounded by edges of valence one in  $B(n)$ . The edges that connect the two boundary curves of such an annulus have alternating valence two and three in  $B(n)$ . Note that  $B(n)$  does not have interior edges.

Given two copies of the triangulation  $B(n)$ ,  $n$  divisible by three, there are two combinatorially different ways of gluing them together along an annulus: either gluing valence two edges to valence two edges and valence three edges to valence three edges, obtaining a double Boerdijk–Coxeter helix with interior edges of valence four and six (we will refer to this as *unshifted gluing*) or identifying the annuli such that valence three edges and valence two edges are identified with one another giving a triangulation with all interior edges of valence five (*shifted gluing*).

Given a triangulation  $K'$  of a surface  $M$  and an  $n \geq 9$  divisible by three, we construct a manifold triangulation  $K$  in the following way: for each triangle of  $K'$  take a copy of  $B(n)$ . Each edge of  $K'$  determines an annulus of the Boerdijk–Coxeter helices corresponding to the two adjacent triangles. Glue the helices along these annuli in the unshifted way. This defines a triangulation  $K$  of  $M \times S^1$ . Every vertex of  $K'$  corresponds to a circle of edges in  $K$ . The valence of these edges is the degree of the vertex in  $K'$ . All other valences in  $K$  are four and six.

*Proof of Theorem 3.15.* Let  $K'$  be a triangulation of  $S^2$  different from the boundary of the simplex. The number  $f_2$  of triangles of  $K'$  is even. Let  $n = \frac{3}{2}f_2$ . Then identifying  $f_2$  copies of  $B(n)$  with a shifted gluing of the annuli corresponding to edges of  $K'$  gives a triangulation  $K$  of  $S^3$ . There is a natural simplicial projection  $f: K \rightarrow K'$ . This map is the Hopf fibration  $S^3 \rightarrow S^2$ .  $\square$

The triangulation  $K$  constructed above is called the *Hopf lift* of  $K'$ . If  $K'$  has  $f$ -vector  $(k+2, 3k, 2k)$  then the Hopf lift of  $K'$  has  $f$ -vector  $(k^2+2k, 7k^2+2k, 12k^2, 6k^2)$ . In particular, since  $k^2 + 2k = (k + 1)^2 - 1$ , a Hopf lift on  $f_0$  vertices exists if and only if  $f_0 + 1 = m^2$  for some  $m \in \mathbb{Z}$ ,  $m \geq 3$ . For  $K' = \partial\Delta_3$  the Hopf lift can be defined but is not a simplicial complex; it is a regular CW complex. Here we need to define  $B(3)$  as a quotient of the helix  $\{[k, k + 1, k + 2, k + 3] \mid k \in \mathbb{Z}\}$  by  $3\mathbb{Z}$ , which is not a simplicial complex. The Hopf lift of the icosahedron is the 600-cell, and the Hopf lift of the octahedron is the 24-cell, where each octahedron of the 24-cell is triangulated by introducing a new edge. These new edges form six circles of length four that are pairwise (Hopf) linked.

Our enumeration of combinatorially positively curved triangulations gives examples of 3-dimensional cone-manifold structures on  $S^3$  with cone angles  $\frac{8\pi}{5}$  and  $\frac{6\pi}{5}$  by inducing the metric of the regular spherical simplex with dihedral angles  $\frac{2\pi}{5}$  on every facet of the triangulation. The volume of the metric can then be computed in a combinatorial way from just the

### 3.4 Three-dimensional examples

number of facets. Hopf lifts give examples of cone-manifold structures where the singular locus consists of Hopf linked circles. This can be generalized to the construction of Seifert fibred triangulation that fiber over a triangulated base orbifold. Below we will present a few more examples of triangulations where the singular locus is a link. The numbers refer to the position of the triangulation in the lexicographically ordered list that can be found online [65].

**Example 3.16** (Two polar trefoils – 4765). In the Boerdijk–Coxeter helix  $B(13)$  the edges of valence one form a single loop that cuts a meridian of  $B(13)$  three times. Two copies of  $B(13)$  can be arranged in such a way with the space between them filled symmetrically with no further vertices such that the valence one edges in  $B(13)$  are precisely the edges of valence four in this triangulation and these edges form trefoil knots. This triangulation has vertex-transitive symmetry and  $f$ -vector  $(26, 130, 208, 104)$ .

**Example 3.17** (A  $(5, 4)$  torus knot – 4768). The 24-cell consists of six Hopf linked loops of four octahedra each (touching vertex to vertex). One can thicken the 24-cell by choosing a loop of four octahedra and replacing them by a loop of four pentagonal bipyramids. In order to close this tessellation of  $S^3$  one has to add octahedra to the complement of the pentagonal bipyramids. Instead of  $5 \cdot 4 = 20$  octahedra one now has  $5 \cdot 5 = 25$  octahedra. The octahedra do not form Hopf linked loops anymore.

This thickened 24-cell (a 29-cell) can be triangulated in a natural way by introducing edges along Hopf fibers to subdivide octahedra into four and pentagonal bipyramids into five tetrahedra. This results in a triangulation of  $S^3$ , where every edge has valence either four or five and edges of valence four are exactly the edges that subdivide octahedra.

There are two loops of valence four edges; one of length five and one of length twenty. The induced subcomplex of the long loop is a torus and the edges of valence four form a  $(5, 4)$  torus knot on it. This triangulation has  $f$ -vector  $(29, 149, 240, 120)$ .

The preceding example suggests a more general definition: let  $K$  be a cell complex homeomorphic to  $S^2$  with a cellular  $\mathbb{Z}/n$ -action that is free away from two antipodal points  $\{\pm x\}$ . We do not require  $x$  or  $-x$  to be vertices but they will necessarily be barycenters of faces. However, we exclude that  $x$  or  $-x$  is the midpoint of an edge. Lifting  $K \setminus \{\pm x\} \cong S^1 \times (0, 1)$  to the universal covering  $\mathbb{R} \times (0, 1)$  induces a cell structure with pointed cells (that is, cells where an interior point is missing) or half-open edges and 2-faces (that is, edges or 2-faces where a vertex is missing). There is a natural, free, cellular action of  $\mathbb{Z}$  on this universal covering, where the subgroup  $n\mathbb{Z}$  is the group of deck transformations extended by lifted  $\mathbb{Z}/n$ -symmetries of  $K \setminus \{\pm x\}$ . Quotienting by some subgroup  $k\mathbb{Z} < \mathbb{Z}$  gives such a cell structure on  $S^1 \times (0, 1)$ , which can be closed off to a cell complex homeomorphic to  $S^2$  by putting  $\{\pm x\}$  back in. This cell complex is called  $\frac{k}{n}$ -unwrapping of  $K$ . We define these

### 3 Combinatorial restrictions on manifold triangulations

unwrappings in the same way for  $K$  a cell complex homeomorphic to  $S^3$  and an embedded unknot  $K'$  such that  $\mathbb{Z}/n$  acts freely and cellularly away from  $K'$ . Here we ask that  $K'$  is a sequence of edges or goes straight from the top to the bottom vertex of an  $n$ -gon.

A CW complex homeomorphic to  $S^2$  with two antipodal hexagons and twelve pentagons is a  $\frac{6}{5}$ -unwrapping of the dodecahedron. A triangular bipyramid is a  $\frac{3}{4}$ -unwrapping of an octahedron. The cone over a  $k$ -gon is a  $\frac{k}{3}$ -unwrapping of the tetrahedron. Extended bipyramids and unwrapping around the a great circle through the suspension points commute, for example, the  $\frac{4}{5}$ -unwrapping of the 600-cell (the extended bipyramid over the icosahedron) is the extended bipyramid over the  $\frac{4}{5}$ -unwrapping of the icosahedron with respect to two antipodal vertices.

Hence Example 3.17 is a  $\frac{5}{4}$ -unwrapping of the Hopf lift of the octahedron with respect to one Hopf circle of valence four. This turns the polar Hopf circle into a loop of five edges of valence four. This loop is fixed by a  $\mathbb{Z}/4$ -action that acts freely on the complement. The corresponding  $\frac{5}{4}$ -unwrapping is the Hopf lift of the pentagonal bipyramid.

**Example 3.18** (3-fold Hopf link with nested  $(1, 2)$  curve – 4770). This triangulation has 36 vertices. The edges of valence four split into four loops, three of which are of length six and form a Hopf link. The fourth loop of valence four edges has length nine. The remaining nine vertices have icosahedral vertex links and can be arranged to lie on a loop of length nine that is parallel to the loop of valence four edges of the same length. In fact, these two loops induce a subcomplex that is homeomorphic to a solid torus with these loops as a  $(4, 2)$ -link on its boundary. The induced subcomplex of the remaining vertices is also a solid torus with the three short loops of valence four edges as a  $(3, 3)$ -link on its boundary. Halfway between these two solid tori is a torus defining the standard Heegaard splitting of  $S^3$ . This triangulation has  $f$ -vector  $(36, 189, 306, 153)$ .

Other interesting examples include regular metric triangulations of a cone-manifold structure on  $S^3$  with singular locus the Borromean rings and cone angles of either  $\frac{6\pi}{5}$  in one instance or  $\frac{8\pi}{5}$  in a second instance and of a cone-manifold structure with cone angle  $\frac{8\pi}{5}$  along a cable knot of the trefoil knot.

Any pseudomanifold with valences of subridges bounded by four is the boundary of a polytope by Theorem 3.6. All triangulations of  $S^2$  are boundaries of polytopes. There are 4761 triangulations of  $S^3$  that are combinatorially positively curved. We will explain one of these triangulations below that is not the boundary complex of a polytope. Very recently, Firsching was able to realize 4759 of the combinatorially positively curved triangulations of  $S^3$  as boundaries of polytopes, even with vertices on the unit sphere (personal communication).



### 3.4 Three-dimensional examples

**Theorem 3.19** (Bokowski and Schuchert [20]). *There is a combinatorially positively curved triangulation  $T$  of  $S^3$  that is not the boundary of a convex 4-polytope.*

Bokowski and Schuchert show that while there is a tiling of  $S^3$  with cubes where two opposite vertices have been truncated, this tiling is not the boundary of a convex 4-polytope. The dual of this tiling is a triangulation of  $S^3$  where edges have valence at most five. We will now describe this triangulation  $T$  (number 2766) in more detail. It was explained along with further properties in [41].

We will explicitly construct  $T$  as a subcomplex of the 5-dimensional crosspolytope. Label the vertices of the 5-dimensional crosspolytope by  $a_0, \dots, a_4, b_0, \dots, b_4$ , where  $a_i$  is opposite to  $b_i$ . The five triangles of the form  $(a_i, a_{i+1}, a_{i+2})$  are a triangulated Möbius strip, as are the triangles  $(b_i, b_{i+2}, b_{i+4})$ , where the indices are always modulo 5. These Möbius strips have five boundary edges and five interior edges. A tetrahedron  $\sigma$  in the 5-dimensional crosspolytope belongs to the subcomplex  $T$  if and only if either a triangle of  $\sigma$  is contained in one of the Möbius strips and the other vertex of  $\sigma$  is in the other Möbius strip or  $\sigma$  has two edges on the two opposite Möbius strips, such that either both are interior edges or both are boundary edges.

The triangulation  $T$  has dihedral symmetry that acts transitively on the set of vertices. Moreover,  $T$  is not weakly vertex decomposable and irreducible. We will define these notions now.

A pure simplicial complex (i.e. one where all facets have the same dimension) is *weakly vertex decomposable* if it is a simplex or if there is a vertex  $v$  such that deleting  $v$  results in a weakly vertex decomposable simplicial complex. A pure simplicial complex is called *vertex decomposable* if it is a simplex or there is a vertex  $v$  such that deleting  $v$  results in a vertex decomposable simplicial complex and the link of this vertex is also vertex decomposable. An edge  $(v, w)$  in a simplicial complex  $K$  is called *contractible* if  $\text{lk}(v, w) = \text{lk}(v) \cap \text{lk}(w)$ . Such an edge can be contracted, and all incident faces collapsed to lower dimensional ones, to still yield a simplicial complex of necessarily the same homotopy type. An edge is contractible if and only if it is not contained in a missing face. A simplicial complex without contractible edges is called *irreducible*.

Trivially, every vertex decomposable triangulation is also weakly vertex decomposable. There is one more implication that holds in general:

**Lemma 3.20** (Klee and Kleinschmidt [56, Sec. 6.2]). *An irreducible triangulation of  $S^d$  different from the boundary of the simplex is not vertex-decomposable.*

The triangulation  $T$  constructed above is set apart from all other 3-dimensional combinatorially positively curved triangulations by these properties. This can be shown by a computer enumeration.

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**Theorem 3.21.** *There is precisely one combinatorially positively curved 3-manifold triangulation that is irreducible apart from the boundary of the simplex (respectively, vertex decomposable, weakly vertex decomposable).*

None of these properties are an obstruction to polytopality. Lockeberg [62] found a simplicial 4-polytope that is not vertex decomposable. In fact, it is irreducible; see Klee and Kleinschmidt [56]. They also give the facet list of Lockeberg’s polytope [56, Sec. 6.3] but with a typo: replace  $aejk$  with  $aehk$ . (This typo was also pointed out in [30].) De Loera and Klee [30] found simplicial  $d$ -polytopes that are not weakly vertex decomposable for every  $d \geq 4$ .

The triangulation  $T$  above is the smallest instance of a triangulation of  $S^3$  that is irreducible apart from the boundary of the simplex (and is the smallest triangulation of  $S^3$  that is not weakly vertex decomposable). This can be verified by a computer enumeration using data of all triangulations of  $S^3$  on at most ten vertices generated by Lutz [65].

**Theorem 3.22.** *Every triangulation of  $S^3$  on less than ten vertices is reducible to the boundary of the simplex and weakly vertex decomposable. Of the 247882 triangulations of  $S^3$  on ten vertices, 1668 are irreducible. Of those, 24 have  $f$ -vector  $(10, 40, 60, 30)$ , and all other irreducible instances have more edges.*

## 3.5 Upper and lower bounds for $f$ -vectors of 3-spheres with valence bounds

We remarked that neighborly instances maximize  $f_3$  (equivalently,  $f_1$  or  $f_2$ ) among all triangulations of  $S^3$  with the same number of vertices  $f_0$ . The  $f$ -vector is minimized for a given number of vertices by a stacked triangulation. These triangulations are not positively curved for  $f_0 \geq 8$ . Here we investigate whether there are classes of triangulations of  $S^3$  minimizing or maximizing the number of facets while constrained to a valence bound. Recall that if  $K'$  is a triangulation of  $S^2$  with  $f$ -vector  $(k+2, 3k, 2k)$  then the extended bipyramid over  $K'$  has  $f$ -vector  $(11k+10, 71k+10, 120k, 60k)$  and the Hopf lift has  $f$ -vector  $(k^2+2k, 7k^2+2k, 12k^2, 6k^2)$ . Two simple calculations show that for extended bipyramids  $f_3 = \frac{60}{11}(f_0-10)$  and for Hopf lifts  $f_3 = 6f_0 - 12\sqrt{f_0+1} + 12$ . An investigation of the census of the 4761 triangulations of  $S^3$  that are combinatorially positively curved shows the following upper and lower bound:

**Theorem 3.23.** *Let  $K$  be a combinatorially positively curved triangulation of  $S^3$  with  $f$ -vector  $(f_0, f_1, f_2, f_3)$ . Then  $\frac{60}{11}(f_0-10) \leq f_3 \leq 6f_0 - 12\sqrt{f_0+1} + 12$ .*

Thus, extended bipyramids minimize  $f_3$  for a given  $f_0$  and Hopf lifts maximize the

number of facets  $f_3$ . The upper and lower bound coincide precisely for the 600-cell with  $f_0 = 120$  vertices.

Does the inequality of Theorem 3.23 remain true for triangulations of  $S^3$  with valence at least five? Since this seems like too much to hope for we conjecture that for such triangulations the average vertex degree  $\bar{z}$  stays bounded. Or equivalently;

**Conjecture 3.24.** *For triangulations of  $S^3$  with valence at least five we have  $f_3 \in \Theta(f_0)$ .*

We attempt to give some evidence in favor of this conjecture. Let a triangulation of  $S^3$  with  $f$ -vector  $(f_0, f_1, f_2, f_3)$  be given and suppose that there are  $t$  facets around every vertex on average. Then by double counting  $4f_3 = tf_0$ . Thus, if  $f_3$  grows superlinearly in  $f_0$  then  $t$  does not stay bounded and vertex links grow with the number of vertices. Now since edges have valence at least five, every vertex link has vertex degrees bounded below by five. Similar ideas to those presented by Elder, McCammond, and Meier [36] in their proof of Thurston's conjecture (see Theorem 3.33) might be applicable here.

## 3.6 Geometry from local combinatorics

We have established that there are only finitely many triangulations of 2- and 3-manifolds with valences bounded by five, and that, moreover, all manifolds admitting such a triangulation are spherical. We still need to investigate which valence bounds do not restrict the topological type of a PL-manifold. For the 3-dimensional case there is the following result:

**Theorem 3.25** (Brady, McCammond, and Meier [21]). *Every closed orientable 3-manifold admits a triangulation with valences four, five, and six.*

We give a much simpler combinatorial proof of Theorem 3.25 and prove a new related version, where we additionally only use three different kinds of vertex links. This improves the previous published bound of Cooper and Thurston [28], that any closed 3-manifold admits a triangulation with five fixed types of vertex links. That these three vertex links suffice was previously observed but not published by Walker.

First we review the classical 2-dimensional case of surface triangulations with bounded vertex degrees. This has close ties to the geometrical structure of the underlying surface. In the following we will use the regular tiling of the plane with six triangles around each vertex. The vertices of this tiling are the *Eisenstein integers*  $a + b\omega$  with  $a, b \in \mathbb{Z}$  and  $\omega = e^{\frac{2\pi}{3}i}$ .

**Lemma 3.26.** *Every  $n$ -gon,  $n \geq 5$ , has a triangulation with new vertices only in the interior that have degree five or six and boundary vertices are contained in at most three triangles.*

*Proof.* For a 5-gon and 6-gon we just need to introduce one new vertex in the interior. For larger  $n$  we define a triangulation of an annulus bounded by an  $n$ -gon on one side and a

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6-gon on the other side that respects the valence condition in the interior. Moreover, if every vertex in the 6-gon is contained in three or four triangles, we can cone it off to obtain a triangulation of the  $n$ -gon as in the statement of the lemma.

Consider the quadrilateral with vertices  $0$ ,  $6$ ,  $(n-6)\omega$ , and  $n+(n-6)\omega$  in the Eisenstein integers. Glue the sides  $(0, (n-6)\omega)$  and  $(6, n+(n-6)\omega)$  together to obtain an annulus. All interior vertices have valence six. The annulus is bounded by a 6-gon with one vertex (corresponding to  $0$  and  $6$ ) contained in four triangles and the other vertices contained in three triangles. The other side of the annulus is an  $n$ -gon with one vertex (corresponding to  $n\omega$  and  $n+(n-6)\omega$ ) contained in two triangles and all other vertices contained in three triangles.  $\square$

**Lemma 3.27.** *A triangle has a triangulation with two new vertices on each edge and one new vertex in the interior, such that the original vertices of the triangle are contained in one triangle, the subdivision vertices on edges are contained in three triangles, and the interior vertex is contained in six triangles.*

*Proof.* This is the triangle with vertices  $0$ ,  $3$ , and  $3+3\omega$  in the Eisenstein integers. The only interior lattice point is then  $2+\omega$ .  $\square$

**Theorem 3.28.** *Every closed surface admits a triangulation with valences five, six, and seven.*

*Proof.* Let  $K$  be an arbitrary triangulation of some closed surface. We modify  $K$  to have degrees five, six, and seven. First triangulate every triangle of  $K$  according to Lemma 3.27 to obtain  $K'$ . All new vertices have degree six. The original vertices of  $K$  are at combinatorial distance three in  $K'$ . Delete vertices of degree three and four in  $K'$  and retriangulate quadrilaterals with a diagonal. This results in a simplicial complex since we subdivided the edges of  $K$ . Moreover, vertex degrees are at least five. We can now delete vertices of degree at least seven and triangulate the resulting holes according to Lemma 3.26. This gives a triangulation with degrees five, six, and seven.  $\square$

Surface triangulations with constant valence are called *equivelar*. The boundary of the icosahedron is an equivelar triangulation of the 2-sphere. The torus can be triangulated with all valences six as a quotient of the regular triangular tiling of the plane. The non-orientable surface with Euler characteristic  $-1$  (sometimes called Dyck's surface) does not admit an equivelar triangulation (in the sense of a simplicial complex). But it does admit a triangulation with valences six and seven. All orientable surfaces, however, admit equivelar triangulations:

**Theorem 3.29** (Combinatorial geometrization). *Every closed orientable hyperbolic surface admits a triangulation with constant valence seven.*

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*Proof.* Construct a triangulation with constant valence seven of the orientable surface of genus two. Such a triangulation can for instance be found in the census [65]. Observe that every orientable surface of genus  $g \geq 2$  is a finite-sheeted cover of this surface and so the triangulation lifts to every closed orientable hyperbolic surface.  $\square$

By inducing the correct metric on every triangle of an equivelar triangulation we obtain as an immediate corollary that every orientable surface admits a constant curvature metric. Then this is also true for the non-orientable quotients.

Thus, a triangulation of constant degree five is spherical, degree six corresponds to Euclidean surfaces, and triangulations with constant degree seven are hyperbolic. By mixing these local combinatorics we can triangulate any surface. We will be interested in a 3-dimensional analogue of this. In Theorem 3.35 we will determine local combinatorics that lead to isotropic geometries in dimension three. That mixing these local combinatorics leads to triangulations of all closed and orientable 3-manifolds, see Theorem 3.32, is a sharpening of Theorem 3.25. We will obtain both Theorem 3.32 and Theorem 3.25 as simple combinatorial corollaries of a first valence bound result for 3-manifolds due to Cooper and Thurston.

**Theorem 3.30** (Cooper and Thurston [28]). *Let  $M$  be a closed orientable 3-manifold. Then  $M$  can be tiled by cubes, such that every edge is in three, four, or five cubes. Moreover, no cube contains two incident edges of valence  $\neq 4$ .*

The barycentric subdivision of this tiling is a triangulation, where vertex links have only five combinatorially different types.

**Corollary 3.31** (Cooper and Thurston [28]). *There are five fixed triangulations of  $S^2$ , such that every closed orientable 3-manifold admits a triangulation with only these 2-spheres as vertex links. Such a triangulation has edge valences bounded by ten.*

We will now improve this corollary and show that in fact three combinatorial types of vertex links suffice to triangulate any closed and orientable manifold. Already in [28] it is pointed out that Walker found such an improvement. Walker's combinatorial refinement (never published) is precisely the one that we give here. Moreover, in this triangulation each triangle has two edges of valence six and one edge of valence three, four, or five.

**Theorem 3.32** (independently Walker). *There are three vertex links, such that every closed orientable 3-manifold can be triangulated with only these vertex link types. They are a fully stacked cube,  $\frac{3}{4}$ -th and  $\frac{5}{4}$ -th of a fully stacked cube. In this triangulation each triangle has two edges of valence six and one edge of valence at most five.*

*Proof.* Let  $M$  be a closed orientable 3-manifold and consider the tiling  $C$  by cubes as in Theorem 3.30. We will modify this tiling to obtain the desired triangulation. First place a

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new vertex at the center of each cube. In any cube each edge and the new subdivision vertex define a triangle. This gives a tiling of  $M$  by square pyramids. Deleting all squares leaves a tiling by octahedra. Now triangulate each octahedron by introducing a new edge connecting two subdivision vertices, that is, the edge is orthogonal to the deleted square.

The edges used to triangulate the octahedra all have valence four. The edges connecting a subdivision vertex to a vertex of the cube have valence six. The original edges of the tiling  $C$  have the same valence as in the tiling.

The vertex link of each vertex at the barycenter of a cube is a fully stacked cube. Original vertices of  $C$  on valence four edges in  $C$  have the same vertex link. The two other vertex links occur for original vertices of  $C$  on edges of valence three or five in  $C$ , respectively.  $\square$

The construction given in the proof above generalizes to the construction of what we will call the *partial dual* of a strongly regular, pure CW complex. Here a CW complex is called *strongly regular* if the closure of each cell is a ball and any two closed cells intersect in precisely one (possibly empty) face. The *dual cell complex*  $C^*$  of a pure, strongly regular,  $d$ -dimensional CW complex  $C$  is defined by putting a vertex at the barycenter of every facet and connecting  $n$  such vertices by a  $k$ -face if the corresponding  $n$  facets intersect in a  $(d-k)$ -dimensional face. Combinatorially, the dual arises by reversing the order of the face poset. The dual complex is again a pure, strongly regular,  $d$ -dimensional CW-complex.

There is a well-defined pairing between primal and dual faces. If the face  $\sigma$  is  $k$ -dimensional, its dual face  $\sigma^*$  is  $(d-k)$ -dimensional. We can naturally identify the bidual  $(C^*)^*$  with the primal  $C$ , and thus for  $\sigma \in C^*$  we have  $\sigma^* \in C$ .

In the following we want to interpolate between the primal and dual complex and define a partial dual. Given two strongly regular CW complexes  $C$  and  $C'$  on disjoint vertex sets their join naturally is a CW complex  $C * C' := \{\sigma * \sigma' : \sigma \in C, \sigma' \in C'\}$ . The join  $C * C'$  is again strongly regular and a set in its face poset is obtained by taking the union of any two sets in the respective posets of  $C$  and  $C'$ .

Let  $C$  be a pure, strongly regular,  $d$ -dimensional CW complex and  $k$  an integer with  $-1 \leq k \leq d$ . The  $k$ -th partial dual  $C^{k*}$  of the complex  $C$  is a certain subcomplex of  $C * C^*$  defined as

$$C^{k*} := \{\sigma * \tau : \sigma \in C^{(d-k-1)}, \tau \in (C^*)^{(k)}, \sigma \subseteq \tau^*\}.$$

We have the special cases  $C^{(-1)*} = C$  and  $C^{d*} = C^*$ . The complex  $C^{0*}$  is obtained from  $C$  by stacking every facet, and dually  $C^{(d-1)*}$  is obtained from  $C^*$  by stacking every facet. In general, we have that  $C^{k*} = (C^*)^{(d-k)*}$ . The complexes  $C$  and  $C^{k*}$  are always PL homeomorphic since the barycentric subdivision of  $C$  is a common subdivision of both complexes.

We will now prove that every closed orientable 3-manifold admits a triangulation with

valences four, five, and six.

*Proof of Theorem 3.25.* Given a closed orientable 3-manifold  $M$  let  $C$  be the tiling by cubes of Theorem 3.30. We assume that edges of valence three in  $C$  are sufficiently far apart in the sense that no two edges in any cube have valence three. (This can be achieved by a cubical subdivision.) Use the same combinatorial modification as in the proof of Theorem 3.32. The only edges of valence three in the resulting triangulations are original edges of  $C$  that are contained in only three cubes. For each such edge choose one adjacent octahedron and instead of triangulating it by introducing a new edge that is orthogonal to the deleted square, triangulate the octahedron by introducing one of the other two possible edges. This increases the valence of all edges in the original square by one. It decreases the valence of four other edges in the octahedron by one. Thus, all edges have valence between four and six. Edges of valence three are sufficiently far apart that valences in any octahedron are changed at most once.  $\square$

We now turn to the question how local combinatorics influence the geometry of a manifold triangulation. We note a result that is somewhat related to our Conjecture 3.24.

**Theorem 3.33** (Elder, McCammond, and Meier [36]). *A closed 3-manifold which admits a triangulation where each triangle has two edges of valence six and the other edge of valence five or six has word-hyperbolic fundamental group.*

**Lemma 3.34.** *There is a spherical (resp. Euclidean, hyperbolic) tetrahedron with two opposite edges with dihedral angles  $\frac{2\pi}{3}$  (resp.  $\frac{2\pi}{4}$ ,  $\frac{2\pi}{5}$ ) and the other edges with dihedral angles  $\frac{2\pi}{6}$ . Edges with the same dihedral angle in a tetrahedron have the same length.*

*Proof.* Projecting the boundary of the 4-simplex onto the 3-sphere gives a tiling by spherical tetrahedra with three facets around each edge. Similarly, Euclidean space can be tiled with four cubes around any edge, and hyperbolic space can be tiled by (hyperbolic) dodecahedra where all edges have valence five. The partial dual  $C^{1*}$  of these tilings  $C$  consists of tetrahedra that have the required metric. The opposite special edges have the same length since these tilings are self-dual. Moreover, a reflection in a 2-face and a reflection in a dual 2-face are symmetries of these triangulations. Thus, edges of valence six have the same length and dihedral angles.  $\square$

**Theorem 3.35.** *Let  $k \in \{3, 4, 5\}$  be fixed. Let  $M$  be a 3-manifold that admits a triangulation  $T$  where each triangle has two edges of valence six and one edge of valence  $k$ . Then  $M$  has geometric structure. It is spherical for  $k = 3$ , Euclidean for  $k = 4$ , and hyperbolic for  $k = 5$ .*

*Proof.* If each triangle consists of edges of valence 6, 6,  $k$ , then each tetrahedron has four edges of valence six and two opposite edges of valence  $k$ . Inducing the spherical, Euclidean,

### 3 Combinatorial restrictions on manifold triangulations

or hyperbolic metric (depending on  $k$ ) given in Lemma 3.34 on each tetrahedron gives the geometric structure. For this notice that if a manifold is locally isometric to a geometric manifold then it inherits the same geometric structure, if the geometry is isotropic.  $\square$

## 3.7 Valence bounds in high dimensions

It is known that any PL-manifold has a triangulation with bounded vertex degrees, where the bound only depends on the dimension. We will give a proof of this folklore theorem below, see Theorem 3.37. This immediately gives some bound on the valence of the triangulation. However, one can expect that there is a bound independent of the dimension of the manifold. We will show that this is in fact true:

**Theorem 3.36.** *Every PL-manifold can be triangulated with valences bounded by nine.*

*Proof.* Consider the dual cell complex of some PL-triangulation, which is a simple cell complex. Triangulate every 2-face as in Lemma 3.26. In any 3-face three 2-faces meet in a vertex. The 2-faces are triangulated such that their boundary vertices are contained in at most three triangles. Thus, in every 3-face vertex degrees are bounded by nine. Stacking every 3-face triangulates the 3-skeleton in such a way that edges are contained in at most nine 3-faces. Continue inductively stacking every face in the  $k$ -skeleton. This defines a triangulation. Since the dual cell complex induces the same PL-structure and the triangulation constructed here subdivides this dual cell complex, this triangulation has the same PL-type.  $\square$

Cooper and Thurston [28] write that “it is known that for any dimension  $n$ , there is a finite set of link types such that every  $n$ -manifold has a triangulation in which the link of each vertex is in this set.” The author was unable to locate a published reference for this claim, so we give a brief proof at this point.

**Theorem 3.37** (folklore). *For every dimension  $d \geq 1$  there are finitely many combinatorial types of vertex links that suffice to triangulate any smooth  $d$ -manifold.*

*Proof.* Given a smooth  $d$ -manifold  $M$ , smoothly embed it into  $\mathbb{R}^{2d}$ . Take a sufficiently fine standard lattice such that  $M$  intersect every  $k$ -cube,  $k \geq d - 1$ , of the corresponding cubulation of  $\mathbb{R}^{2d}$  transversely in a ball of dimension  $(k - d)$ . This induces a regular cell complex structure on  $M$ , where vertices correspond to  $d$ -faces of the cubulation. Vertex degrees of this cell complex are bounded by  $2^d$ . The faces of this cell complex have at most as many vertices as  $d$ -faces in a  $2d$ -cube, that is,  $\binom{2d}{d}2^d$ . Thus, the barycentric subdivision is a triangulation of  $M$  with vertex degrees bounded above by a number that only depends on the dimension  $d$ .  $\square$



# 4 Geometric structures and Shephard's conjecture

## Abstract

The boundary of a polytope has a Euclidean structure away from its subridges. We use the corresponding developing map to unfold the boundary of a polytope in Euclidean space without overlap. This was previously done by Miller and Pak [73] via the exponential map. Our approach is very similar but somewhat more flexible.

We refer to Ziegler [106] for the basics of polytope theory. There are essentially two ways to visualize a  $(d + 1)$ -dimensional polytope in  $\mathbb{R}^d$ , either by means of a Schlegel diagram or by a net of the polytope. A *Schlegel diagram* is obtained by projecting the polytope onto a facet through a point beyond but near to the facet. A net is constructed by cutting the polytope along certain ridges and unfolding the resulting polyhedral space in  $\mathbb{R}^d$  without overlap. Thus, it is a model which can be glued to give back the polytope.

More precisely, a *net* of a  $(d + 1)$ -polytope  $P$  is a facet-ridge tree  $T$  (that is, a tree in the 1-skeleton of the dual polytope) and for every facet  $\sigma \in P$  an isometry  $f_\sigma: \sigma \rightarrow \mathbb{R}^d$ , such that  $f_\sigma(\sigma) \cap f_\tau(\tau) = f_\sigma(\tau \cap \sigma)$  for every pair of facets  $(\sigma, \tau) \in T$  in the facet-ridge tree, and  $f_\sigma(\sigma) \cap f_\tau(\tau)$  has dimension at most  $d - 2$  for any two distinct facets  $\sigma$  and  $\tau$  that are not joined by an edge in  $T$ . This says that a net is a  $d$ -dimensional polyhedral space, isometrically embedded into  $\mathbb{R}^d$ , that has facetwise the same metric as the polytope and preserves incidences among facets along a dual tree (and no other incidences). It was conjectured by Shephard [86] that every 3-polytope has a net. Shephard attributes this conjecture to Albrecht Dürer [34]. This conjecture and its higher-dimensional analogues have remained unsolved. It is known to be false for non-convex (but star-shaped) 3-polyhedra with convex faces; see Tarasov [92] and Grünbaum [47].

Every convex 3-polytope can be affinely dilated in such a way that it has a net; see Ghomi [44]. Thus, there are no combinatorial obstructions to Shephard's conjecture. Miller and Pak [73] showed that there are no metric obstructions in any dimension, that is, the

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boundary of any convex  $d$ -polytope has a (geodesic) subdivision that admits a net. Their results hold more generally for Euclidean cone-manifolds with cone-angles smaller than  $2\pi$ .

Here we show that it is natural to unfold a subspace of the universal covering of a 3-polytope without vertices. In fact, this approach works in the same way in higher dimensions by deleting the subridges of the polytope. We show that results of this kind are most naturally obtained from a geometric structure and its corresponding developing map.

### 4.1 Geometric structures

We refer to Thurston [94] for an introduction to geometric structures and the developing map. Let  $Y$  be a Riemannian manifold and  $G$  a group of isometries of  $Y$  such that each isometry  $g \in G$  is uniquely determined by its restriction to any non-empty open set: if  $U \subseteq Y$  is non-empty and open, and  $g, h \in G$  such that  $g \cdot u = h \cdot u$  for all  $u \in U$ , then  $g = h$ .

Suppose the Riemannian manifold  $X$  is locally isometric to  $Y$  in such a way that every change of coordinates is a restriction of an isometry in  $G$ , that is, every  $x \in X$  has a neighborhood  $U_x \subseteq X$  and a function  $\phi_x : U_x \rightarrow Y$  that preserves distances, and if for  $x, y \in X$  these neighborhoods  $U_x$  and  $U_y$  are not disjoint, then there is a  $g \in G$ , such that  $\phi_x \circ \phi_y^{-1}|_{\phi_x(U_x \cap U_y)} = g|_{\phi_x(U_x \cap U_y)}$ . The Riemannian manifold  $X$  is said to have a *geometric structure modelled on  $(G, Y)$* , and a maximal collection of such coordinate neighborhoods  $\{(U, \phi)\}$  is called a *geometric structure*.

Let  $x_0 \in X$  and find a coordinate neighborhood  $U$  of  $x_0$  with an isometry  $\Phi$  to some open set in  $Y$ . Let  $\gamma$  be a path in  $X$  starting at  $x_0$ . We can extend  $\Phi$  along  $\gamma$ . Given a point  $x$  on  $\gamma$  and a coordinate neighborhood  $V$  of  $x$  with an isometry  $\Psi$  to some open set in  $Y$ , such that  $U \cap V \neq \emptyset$ , the coordinate transformation  $\Phi \circ \Psi^{-1}$  is given by the restriction of some  $g \in G$ . We extend  $\Phi$  by setting  $\Phi(x) = g \cdot \Psi(x)$ . Finitely many such continuation steps suffice, since the image of  $\gamma$  is compact. Since continuation along a homotopic path gives the same result, this defines a well-defined map  $\text{dev} : \tilde{X} \rightarrow Y$  on the universal covering space  $\tilde{X}$  of  $X$  called the *developing map*. This map depends on the point  $x_0$  and isometry  $\Phi$ , but it is unique up to multiplying with an element in  $G$ . By construction the developing map is a local isometry. Hence, also  $\tilde{X}$  has geometric structure modelled on  $(G, Y)$ . A *geodesic* in a cone-manifold is a path that is locally a shortest path. Every local isometry maps geodesics to geodesics.

**Theorem 4.1.** *Let  $X$  be a simply connected Riemannian manifold with geometry modelled on  $(G, \mathbb{R}^d)$  for some group of Euclidean isometries  $G$ . Suppose there is a set  $U \subseteq X$  and a point  $x_0 \in U$  such that any other point in  $U$  is connected to  $x_0$  via at least one geodesic. Then  $\text{dev}|_U : U \rightarrow \mathbb{R}^d$  is injective.*

If  $X$  is complete then  $U = X$  by the Hopf–Rinow theorem.

*Proof.* Consider the developing map. Suppose  $\text{dev}(x) = \text{dev}(x')$  for some  $x, x' \in U$ ,  $x \neq x'$ . There are geodesics  $\gamma$  and  $\gamma'$  in  $X$  connecting  $x_0$  to  $x$  and  $x'$ , respectively. Since  $\text{dev}$  is a local isometry  $\text{dev} \circ \gamma$  and  $\text{dev} \circ \gamma'$  are geodesics in  $\mathbb{R}^d$  that intersect in  $\text{dev}(x_0)$  and  $\text{dev}(x) = \text{dev}(x')$ . Geodesics in  $\mathbb{R}^d$  cannot contain loops and if they intersect in two distinct points they are equal between these two points. Thus, w.l.o.g. the image of  $\text{dev} \circ \gamma$  is contained in the image of  $\text{dev} \circ \gamma'$ . Then  $\text{dev}$  is not even injective in any neighborhood of  $x_0$ , which is a contradiction to the definition of developing map.  $\square$

**Remark 4.2.** In the proof of the previous theorem we only use that geodesics intersect in at most one point in Euclidean space and a geodesic does not revisit any point, that is, there are no closed geodesics. Thus, if the geometric structure on  $X$  is induced by a space with these properties, Theorem 4.1 holds in the same way. In particular, it holds for simply connected hyperbolic manifolds.

**Theorem 4.3.** *Let  $X$  be a simply connected Riemannian manifold with geometry modelled on  $(G, S^d)$  for some group of spherical isometries  $G$ . Suppose there is a set  $U \subseteq X$  and a point  $x_0 \in U$  such that any other point in  $U$  is connected to  $x_0$  via at least one geodesic of length  $< \pi$ . Then  $\text{dev}|_U: U \rightarrow S^d$  is injective.*

*Proof.* The proof is the same as for Theorem 4.1, now observing that any two geodesics in  $S^d$  intersect each other in points that are at distance  $< \pi$  along both geodesics coincide for the length of one of the geodesics.  $\square$

## 4.2 Geodesics in cone-manifolds

Let  $Y$  be a topological space, and let a free, properly discontinuous action by the group  $G$  acting by homeomorphisms on  $Y$  be given. Then a path-connected subspace  $X \subseteq Y$  is called a *fundamental domain* if  $\{g \cdot X : g \in G\}$  is a partition of  $Y$ .

We briefly repeat the definition of cone-manifold for the case of Euclidean structure. Let  $M$  be a  $d$ -dimensional topological manifold with a triangulation  $T$  such that  $M$  has a metric  $d$  that is induced by the Euclidean metric on every facet. Then  $(M, d)$  is called *Euclidean cone-manifold*. Points that do not have a neighborhood isometric to a ball in  $\mathbb{R}^d$  are called *singular*. The set of singular points is a union of subridges in  $T$  called the *singular locus*. Let  $x$  be a point in the singular locus of  $M$ . Then a neighborhood of  $x$  is isometric to a cone over some metric  $(d - 1)$ -sphere.

Spherical or hyperbolic cone-manifolds are defined similarly by being glued from spherical or hyperbolic simplices. Deleting the singular locus from a Euclidean cone-manifold gives

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a Riemannian manifold with Euclidean structure. (This remains true for any isotropic geometry.) This manifold is incomplete if the singular locus is non-empty.

Let  $X$  be a  $d$ -dimensional Euclidean cone-manifold with singular locus  $S$ . Let  $M = X \setminus S$  with universal covering  $p: \widetilde{M} \rightarrow M$ . Then  $X$  is called *developable* if the developing map  $\text{dev}: \widetilde{M} \rightarrow \mathbb{R}^d$  is injective on a fundamental domain of  $\widetilde{M}$  with respect to the usual action of  $\pi_1(M)$  by deck transformations.

To prove that Euclidean cone-manifolds with cone angles  $< 2\pi$  are developable, we will need two lemmas.

**Lemma 4.4** (Burago, Burago, Ivanov [24, Prop. 2.5.19]). *Let  $X$  be a compact metric space. If  $x, y \in X$  can be connected by a path of finite length, they can be connected by a shortest path.*

**Lemma 4.5** (Cooper, Hodgson, Kerckhoff [27, page 63]). *In a cone-manifold with all cone-angles less than  $2\pi$ , if the interior of a geodesic contains a point in the singular locus, then the entire geodesic is contained in a single stratum of the singular locus.*

**Theorem 4.6.** *Let  $X$  be a compact  $d$ -dimensional Euclidean cone-manifold with cone-angles  $< 2\pi$ . Then  $X$  is developable.*

*Proof.* Let  $M$  be the Euclidean manifold obtained from  $X$  by deleting the singular locus. Let  $x_0 \in \widetilde{M}$  be some point in the universal covering space of  $M$  and define  $U = \{x \in \widetilde{M} : \text{there is a geodesic from } x_0 \text{ to } x\}$ . Then according to Theorem 4.1 the developing map in  $x_0$  is injective on  $U$ . We will show that  $U$  contains a fundamental domain of  $M$ .

Let  $p: \widetilde{M} \rightarrow M$  be the covering map, and let  $y \in M$  be an arbitrary point. There is a shortest path  $\gamma$  in  $X$  connecting  $p(x_0)$  to  $y$  by Lemma 4.4. The path  $\gamma$  is even a geodesic in  $M$  by Lemma 4.5 and thus lifts to a geodesic in  $\widetilde{M}$ . The endpoints of these geodesics are a fundamental domain of  $M$  in  $\widetilde{M}$ . The set of endpoints is path-connected since the geodesics are closed under taking initial segments and emanate from  $p(x_0)$ . For every point in  $M$  there is exactly one endpoint of a chosen geodesic in  $\widetilde{M}$ . Thus, this set induces a partition of  $\widetilde{M}$  under the action of  $\pi_1(M)$ .  $\square$

**Remark 4.7.** In fact, here we used special fundamental domains to construct non-overlapping unfoldings in the case where cone angles are  $< 2\pi$ : let  $\Gamma$  be a set of geodesics in  $M$  emanating from  $p(x_0) \in M$ , such that every point is the endpoint of exactly one geodesic and any initial segment of a geodesic in  $\Gamma$  is again in  $\Gamma$ . The geodesics  $\Gamma$  lift in a unique way to the covering  $\widetilde{M}$  emanating from  $x_0$ . The set of points that can be reached by these lifted geodesics is a fundamental domain and the developing map is injective on them.

### 4.3 Unfolding polytopes

Specializing to convex polytopes the corollary below is a direct consequence of Theorem 4.1.

**Corollary 4.8.** *Let  $P$  be a  $(d+1)$ -polytope,  $M = \partial P \setminus P^{(d-2)}$ , and  $x_0 \in \widetilde{M}$ . Let  $U = \{x \in \widetilde{M} : \text{there is a geodesic from } x_0 \text{ to } x\}$ . Then the developing map  $\text{dev}: \widetilde{M} \rightarrow \mathbb{R}^d$  in  $x_0$  is injective on  $U$ .*

The unfolding constructed here unfolds a point  $x$  in the boundary of the polytope multiple times if there are multiple geodesics connecting  $x_0$  to a lift of  $x$  in the universal covering  $\widetilde{M}$ . If we want to unfold every point only once then we would need to choose a specific set of geodesics  $\Gamma$  as in Remark 4.7. For example, we can obtain such a set  $\Gamma$  by lifting unique shortest paths in  $M$  to  $\widetilde{M}$ . This gives the theorem of Miller and Pak [73]. We have, however, additional flexibility as the example below shows.

**Example 4.9.** Choosing  $\Gamma$  to be the set of shortest paths from a fixed point  $x_0$  does not give a net for the standard 3-cube. However, there is a set of geodesics  $\Gamma$  that gives a net: let  $\sigma$  be the facet of the cube that contains  $x_0$  and let  $\tau$  be some adjacent facet; for every point  $y$  in an adjacent facet of  $\sigma$  let the shortest path connecting  $x_0$  and  $y$  be in  $\Gamma$  (or rather the unique lift to the universal covering thereof), and for a point  $y$  in the opposite facet of  $\sigma$ , let the (lift of the) geodesic from  $x_0$  to  $y$  that only goes through  $\sigma$ ,  $\tau$ , and  $-\sigma$  be in  $\Gamma$ .

We can generalize this observation. The space  $M = \partial P \setminus P^{(d-2)}$  is tiled by the facets and ridges of  $P$  and hence the universal covering  $\widetilde{M}$  inherits a tiling by (non-closed) polytopes. If the universal covering contains a star-shaped, polyhedral fundamental domain, then  $P$  has a (star-shaped) net.

**Corollary 4.10.** *Let  $P$  be a  $(d+1)$ -polytope and  $M = \partial P \setminus P^{(d-2)}$  with universal covering  $\widetilde{M}$ . If there is a point  $x_0 \in \widetilde{M}$  such that  $U = \{x \in \widetilde{M} : \text{there is a geodesic from } x_0 \text{ to } x\}$  contains a polyhedral fundamental domain, then  $P$  has a star-shaped net in  $\mathbb{R}^d$ .*

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